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Probability of Error Analysis Using a Gauss-Chebyshev Quadrature Rule

Thesis directed by Professor Rodger E. Ziemer

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PROBABILITY OF ERROR ANALYSIS

USING A GAUSS-CHEBYSHEV QUADRATURE RULE

by

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DEDICATION

In memory of my grandfather who inspired me to follow my dreams and do the best in everything I do.

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Figure

CHAPTER I

INTRODUCTION

1.1 Problem Statement

The ability to analyze error probability is an important step in designing and implementing a communication system. In some cases, this analysis can be very easy while in others it can be very difficult. There are numerous techniques available to analyze error performance. The most desirable is to find an analytic solution for the error probability. In many cases, analytic solutions are extremely difficult to obtain and hardware implementation or computer-aided simulation may be required to facilitate analysis [1]. Additionally, numerical and residue evaluation techniques may be used in conjunction with the analytic and computer-aided methods. The method of choice is the one that obtains accurate error performance results quickly, easily, and economically.

One method to analyze error performance of a communication system is proposed by Biglieri, Caire, Taricco, and Ventura-Traveset in [2] and [3]. The method makes use of the two-sided Laplace transform of the probability density function (PDF) of a decision statistic. A Gauss-Chebyshev quadrature rule is then applied to the inverse of this Laplace transform to calculate the error probability. This thesis will center on fully explaining the derivation of this method. Additionally, the method is verified by analyzing the error performance of several communication systems of varying

complexity. A major objective is to determine and explain how to optimally implement the method in order to reduce computational time while maintaining numerical accuracy. As shown in this thesis, the method has potentially broad ranging utility, especially when a closed form solution for the error probability is not obtainable using other techniques.

1.2 Literature Search

As stated, there are many techniques used to analyze error probability in communication systems. These can be broadly grouped under finding analytic solutions, computer-aided analysis, and hardware implementation. Furthermore, numerical and residue techniques may be used along with these broad methods.

Finding an analytic solution for the error performance is the most desirable technique. The resulting solution may have a closed form or include an integral requiring numerical evaluation. As stated in [1], analytic analysis is very useful in the early stages of system design for exploring relationships between design parameters and system performance. However, such analysis is usually based on simplified models and analytic results are generally harder to obtain as system complexity increases. The problem is further compounded if one is required to compare the error performances of several systems.

There are numerous examples of employing analytic methods. One need only look in a digital communications book to find many such examples. Another example is that of Lindsey's investigation into error probabilities for binary and M-ary signals in a Rician fading channel in [4]. Simon and Alouini also find analytic results for error performance in generalized fading channels in [5]. To accomplish this analysis, they

make use of alternative forms of the Marcum Q-function described in [6] and [7]. This shows how the numerical Q-function, which is an integral with no closed form, is manipulated to obtain analytic results.

The next broad category to evaluate error performance is computer-aided analysis. This could range from simple implementation of analytic equations to a full-blown waveform level simulation [1]. Computer simulation methods include Monte-Carlo simulation, importance sampling, extreme value theory analysis, tail extrapolation, quasianalytic techniques, and others. Using simulation allows analysis of a system to any desired level of detail. Furthermore, a multitude of complex systems can be considered with relative ease and in less time than through the other broad methods.

Several examples of how simulation is used to determine error performance were examined. Chen, Lu, Sadowsky, and Yao explain a strategy of how to employ importance sampling in [8]. Also, in [9], Stadler and Roy describe how to use adaptive importance sampling to decrease computational time. Finally, Tranter and Schneider introduce another simulation technique called Partial Sum of Products (ParSOP) to simulate digital communication systems in nonlinear channels in [10].

Hardware implementation is another possible way to determine the error performance of a system. Basically, this method involves actually building a system and measuring its error performance over the communication channel being used. This could also simply involve measuring channel characteristics at the desired frequencies, bandwidths, or other design parameters possibly affected by the channel [1]. Two examples of channel characterization are shown in [11] and [12]. Results using this method are often accurate and reliable. However, the drawback is that it is expensive and

not very flexible. This technique is useful during the later stages of the design process when there are fewer choices to consider.

In reality, a combination of the broad methods of error performance analysis is required when actually designing a system. Analytic analysis is the basis of all the methods. This is because the theoretical attributes of a system must be understood through at least some analytic analysis before one can model and actually implement the system. It may not be possible to account for all of the actual sources of errors in a system by analytic analysis, but it must be the basis of any design process. Once a basic understanding of a system is in hand, computer simulation can be conducted to fully investigate all possible sources of errors. This simulation might involve using actual measured attributes of a system under consideration or measured channel characteristics. Finally, simulated waveforms may be used in testing the resulting hardware implementation and to help verify the validity of the computer simulation used in the design process.

As previously stated, numerical and residue integration techniques are commonly used in error performance analysis using the analytic or computer-aided analysis methods. Using the Marcum Q-function could be considered a "numerical technique" since it is an integral that is not expressible in closed form and must be numerically computed. Additionally, the application of a Gauss-Quadrature rule in the method under consideration in this thesis can be considered a numerical technique. A residue integration technique could be used in place of this Gauss-Quadrature rule to evaluate error performance. This thesis centers on using the Gauss-Quadrature rule to aid in finding analytic error performance results and implementing them using a computer.

Thus, it is shown how a combination of analytic and computer-aided methods, along with a numerical technique, is used to evaluate error performance in communication systems.

1.3 Thesis Overview

Chapter 1 has introduced the method under consideration along with a short description of the broad categories of error performance analysis methods.

Chapter 2 explains the method derivation. First, the relationship between the probability of error and the inverse of the two-sided Laplace transform is established. This is followed by application of the Gauss-Chebyshev quadrature rule.

Chapter 3 implements the method to analyze error performance in several communication system models. Each system is fully described, a theoretical probability of error is derived, and the method is applied. This is followed by implementing the method using a computer and comparing the resulting error probabilities to the theoretical results.

Chapter 4 discusses the method implementation. Suggestions for implementation are made as an aid to generating accurate results with as little computational time as possible.

Finally, Chapter 5 is the concluding chapter where recommendations for further research are made.

CHAPTER II

METHOD DEVELOPMENT

This chapter introduces the method under consideration that is used to obtain error probability in communication systems. An explanation of how the inverse of the two-sided Laplace transform of the PDF of a decision statistic is related to error probability is first presented. After establishing this relationship, the application of the Gauss-Chebyshev quadrature rule to numerically integrate this inverse transform is discussed.

2.1 Probability of Error using the Two-Sided Laplace Transform

A digital communication system has a finite signal set used to convey information through a communication channel. At the receiver, a received signal is compared to all possible transmitted signals to generate decision metrics. A decision metric is basically the probability that a received signal resulted from one of the possible transmitted signals. Decision metrics are determined for each signal of the signal set. The largest decision metric is used to decide which signal was transmitted, hopefully with high probability.

The method under consideration begins by defining the metric $m(y,x)$ where x is the signal that was transmitted and y is the signal received. Again, a decision is made at the receiver by selecting from all possible transmitted signals the one that has the maximum metric [2]. Assuming that x was sent, the pairwise error probability (PEP) is

$$P\{x \rightarrow x'\}. \tag{2.1}$$

Basically, the PEP is the probability that when x is transmitted and y received the metric $m(y,x')$ is greater than $m(y,x)$. The variable x' represents all other possible transmitted signals other than x. Next, the random variable Y is defined as

$$Y = m(y,x) - m(y,x') \tag{2.2}$$

where it is obvious that the signal is received in error if Y is negative. Therefore, the PEP is related to the random decision metric by

$$P\{x \rightarrow x'\} = P\{Y < 0\}. \tag{2.3}$$

Thus, as shown in (2.3), finding the error probability of a communication system reduces to finding the probability that the random decision metric Y is negative. Since the decision metric, Y, is a random variable it will have a PDF associated with it that defines how the variable is distributed. The method continues by taking the two-sided Laplace transform of this PDF, $p_Y(y)$, resulting in

$$\Phi_Y(s) = E(e^{-sY}) = \int_{-\infty}^{\infty} e^{-sy} p_Y(y)dy \tag{2.4}$$

where $s = \alpha + j\omega$.

Based on Laplace-transform theory and (2.4), there are several important observations to be made before continuing the method. First, the integral of (2.4) has a region of convergence (ROC) that is a vertical strip in the complex s-plane. This is analogous to the derivation in [14] concerning analysis of system stability. The ROC comes from the fact that the two-sided Laplace transform, $X(s)$, of some function, $x(u)$, defined by

$$X(s) = \int_{-\infty}^{\infty} x(u)e^{-su}du, \tag{2.5}$$

converges absolutely if

$$\int_{-\infty}^{\infty} \left| x(u) e^{-su} \right| du < \infty \tag{2.6}$$

where $s = \alpha + j\omega$. The requirement for convergence from (2.6) can be rewritten as

$$\int_{-\infty}^{\infty} |x(u)| e^{-\alpha u} du < \infty . \tag{2.7}$$

Correspondingly, the Laplace transform converges absolutely if a real positive number A exist such that for some real b and c

$$|x(u)| \leq \begin{cases} Ae^{cu}, & u > 0 \\ Ae^{bu}, & u < 0 \end{cases} . \tag{2.8}$$

By breaking the integral (2.5) into the two parts where $u > 0$ and $u < 0$, it can be shown that the singularities of $X(s)$ arising from the portion of $x(u)$ for $u > 0$ will lie to the left of the line $s = c$. Similarly, the singularities of $X(s)$ from the part of $x(u)$ for $u < 0$ will lie to the right of the line $s = b$. Therefore, it is known that the ROC of (2.5) will be some vertical strip in the s-plane. Thus, if the conditions in (2.8) are met then the ROC is defined by

$$c \leq \alpha \leq b . \tag{2.9}$$

Again, consider the Laplace transform, $X(s)$, of some function $x(u)$. The reasoning that follows is similar to that concerning bounded-input, bounded-output (BIBO) stability in systems analysis. If

$$\int_{-\infty}^{\infty} |x(u)| du < \infty \tag{2.10}$$

then any singularities resulting from the portion of $x(u)$ for $u > 0$ must lie to the left of the $j\omega$ axis in the s-plane [13,14]. Similarly, the singularities from the part of $x(u)$ for $u < 0$ must lie to the right of the $j\omega$ axis. Therefore, from (2.9), the ROC of the two-sided Laplace transform of some function $x(u)$ is a vertical strip in the s-plane that includes the $j\omega$ axis if $x(u)$ meets the requirement of (2.10).

In the method under consideration, the two-sided Laplace transform of the PDF associated with the decision metric Y shown in (2.4) is desired. From random variable theory we know that the PDF of the random decision metric Y has the property

$$p_Y(y) \geq 0 \tag{2.11}$$

for $\infty \ < y < \infty$. This leads to the relationship

$$|p_Y(y)| = p_Y(y), \quad -\infty \leq y \leq \infty \tag{2.12}$$

and

$$\int_{-\infty}^{\infty} p_Y(y) dy = \int_{-\infty}^{\infty} |p_Y(y)| dy = 1. \tag{2.13}$$

Therefore, (2.13) along with (2.10) imply that the two-sided Laplace transform of the PDF $p_Y(y)$ has a ROC which is a vertical strip in the s-plane that includes the imaginary axis. This vertical strip is bounded by the singularities of $\Phi_Y(s)$ that are closest to the imaginary axis. The possible i singularities of $\Phi_Y(s)$ are denoted in [2] as s_i. Therefore, the ROC is $\alpha_1 < \alpha < \alpha_2$ where

$$\alpha_1 = \max_{\mathrm{Re}[s_i]<0} \left(\mathrm{Re}[s_i]\right) \tag{2.14}$$

and

$$\alpha_2 = \min_{\mathrm{Re}[s_i]>0} \left(\mathrm{Re}[s_i]\right). \tag{2.15}$$

It is evident from (2.14) and (2.15) that α_1 will have a negative value and α_2 a positive one.

The next step in the method development involves applying integration by parts to (2.4). To accomplish this, let

$$u(y) = e^{-sy}, \tag{2.16}$$

$$u'(y) = \frac{du(y)}{dy} = -se^{-sy}, \tag{2.17}$$

$$v'(y) = p_Y(y), \tag{2.18}$$

and

$$v(y) = \int_{-\infty}^{y} p_Y(\beta)d\beta. \tag{2.19}$$

The assignment of (2.19) is, by definition, the cumulative distribution function (CDF) of the random variable Y [15]. Therefore, $v(y)$ is

$$v(y) = F_Y(y). \tag{2.20}$$

Next, by applying the integration by parts formula we get

$$\begin{aligned}\Phi_Y(s) &= \int u(y)v'(y)dy = u(y)v(y) - \int v(y)u'(y)dy \\ &= e^{-sy}F_Y(y)\Big|_{-\infty}^{\infty} + s\int_{-\infty}^{\infty} F_Y(y)e^{-sy}dy \\ &= e^{-s\infty}F_Y(\infty) - e^{s\infty}F_Y(-\infty) + s\int_{-\infty}^{\infty} F_Y(y)e^{-sy}dy\end{aligned} \tag{2.21}$$

However, random variable theory says that $F_Y(\infty) = 1$ and $F_Y(-\infty) = 0$. Assuming that $\alpha = \mathrm{Re}(s) > 0$ results in $e^{-s\infty} = 0$. Therefore, the first term in the last line of (2.21) goes to zero. Finally, assume that as $y \to -\infty$, $F_Y(y)$ goes to zero faster than e^{-sy} goes to ∞. Hence, the middle term on the last line of (2.21) goes to zero. The final result is

$$\Phi_Y(s) = s\int_{-\infty}^{\infty} F_Y(y)e^{-sy}dy. \tag{2.22}$$

The inverse Laplace transform is given by

$$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}dt. \tag{2.23}$$

Applying the inverse Laplace transform to (2.22) results in

$$F_Y(y) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \Phi_Y(s)e^{sy}\frac{ds}{s} \quad 0 < c \le \alpha_2. \tag{2.24}$$

Applying the fact that $P\{Y<0\} = F_Y(0)$ yields

$$P\{Y<0\} = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \Phi_Y(s)e^{s0}\frac{ds}{s} = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \Phi_Y(s)\frac{ds}{s}. \tag{2.25}$$

The error probability of (2.25) can be evaluated directly through residues. However, in many cases this may be very tedious. Therefore, in this thesis we apply the Gauss-Chebyshev quadrature rule to numerically integrate (2.25) and obtain the error probability.

2.2 Applying the Gauss-Quadrature Rule

As previously mentioned, this paper centers on evaluation of (2.25) through use of a Gauss-quadrature rule. To apply this rule, we first note that the value of $P\{Y<0\}$ should

be real; therefore, the real part of (2.25) is sought [2]. The real part of a complex number after complex division is dictated by

$$\mathrm{Re}\left\{\frac{a+bi}{c+di}\right\}=\mathrm{Re}\left\{\frac{a+bi}{c+di}\frac{c-di}{c-di}\right\}=\mathrm{Re}\left\{\frac{ac+bd}{c^2+d^2}+i\frac{bc-ad}{c^2+d^2}\right\}=\frac{ac+bd}{c^2+d^2}. \quad (2.26)$$

Substituting $s = c + j\omega$ and $ds = jd\omega$, and applying (2.26) to (2.25) results in the real part of $P\{Y<0\}$ given by

$$\begin{aligned} P\{Y<0\} &= \mathrm{Re}\left\{\frac{j}{2\pi j}\int_L^U \frac{\Phi_Y(c+j\omega)}{c+j\omega}d\omega\right\} \\ &= \mathrm{Re}\left\{\frac{1}{2\pi}\int_L^U \frac{\mathrm{Re}\{\Phi_Y(c+j\omega)\}+\mathrm{Im}\{\Phi_Y(c+j\omega)\}}{c+j\omega}\frac{c-j\omega}{c-j\omega}d\omega\right\}. \quad (2.27) \\ &= \frac{1}{2\pi}\int_L^U \frac{c\,\mathrm{Re}[\Phi_Y(c+j\omega)]+\omega\,\mathrm{Im}[\Phi_Y(c+j\omega)]}{c^2+\omega^2}d\omega \end{aligned}$$

The new upper and lower limits of integration are

$$U=\left.\frac{s-c}{j}\right|_{s=c+j\infty}=\frac{c+j\infty-c}{j}=\frac{j\infty}{j}=\infty \quad (2.28)$$

and

$$L=\left.\frac{s-c}{j}\right|_{s=c-j\infty}=\frac{c-j\infty-c}{j}=\frac{-j\infty}{j}=-\infty. \quad (2.29)$$

Now the change of variables

$$\omega=c\sqrt{1-x^2}\big/x, \quad (2.30)$$

with $d\omega$ given by

$$d\omega=\frac{-c}{x^2\sqrt{1-x^2}}dx, \quad (2.31)$$

is applied to (2.27) to yield

$$P\{Y<0\}=\frac{1}{2\pi}\int_L^U\left[\frac{c\operatorname{Re}\{q\}+\omega\operatorname{Im}\{q\}}{c^2+\omega^2}\right]\frac{-c\,dx}{x^2\sqrt{1-x^2}}$$
$$=\frac{1}{2\pi}\int_L^U\left[c\operatorname{Re}\{q\}+\frac{c\sqrt{1-x^2}}{x}\operatorname{Im}\{q\}\right]\frac{-c\,dx}{\sqrt{1-x^2}\left(x^2c^2+c^2-c^2x^2\right)} \tag{2.32}$$
$$=-\frac{1}{2\pi}\int_L^U\left[\operatorname{Re}\{q\}+\frac{\sqrt{1-x^2}}{x}\operatorname{Im}\{q\}\right]\frac{dx}{\sqrt{1-x^2}}$$

where $q = \Phi_Y(c + j\omega)$. To help explain the upper and lower limits for the integral of (2.32), the variable transformation of (2.30) is plotted in Figure 2.1. As shown in (2.30), there is a discontinuity at $x = 0$. Therefore, disregarding the integrand for now, the integral is broken into the two parts

$$\int_{-\infty}^{\infty}d\omega=\int_{-\infty}^{0^-}d\omega+\int_{0^+}^{\infty}d\omega\,. \tag{2.33}$$

As seen in Figure 2.1, the range of ω over $[-\infty, 0^-)$ corresponds to the range on x being $(0^-, -1)$. Similarly, the ω range $(0^+,\infty]$ is commensurate with the range on x of $(0^+, 1)$. Therefore, after the change of variables, the resulting integration is

$$\int_{-\infty}^{\infty}d\omega=\int_{0^-}^{-1}\frac{-c}{x^2\sqrt{1-x^2}}dx+\int_{1}^{0^+}\frac{-c}{x^2\sqrt{1-x^2}}dx\,. \tag{2.34}$$

Applying the fact that

$$\int_a^b f(x)dx=-\int_b^a f(x)dx \tag{2.35}$$

to (2.34) results in

$$\begin{aligned}\int_{-\infty}^{\infty} d\omega &= \int_{0^-}^{-1} \frac{-c}{x^2\sqrt{1-x^2}}dx + \int_{1}^{0^+} \frac{-c}{x^2\sqrt{1-x^2}}dx \\ &= \int_{-1}^{0^-} \frac{c}{x^2\sqrt{1-x^2}}dx + \int_{0^+}^{1} \frac{c}{x^2\sqrt{1-x^2}}dx . \\ &= \int_{-1}^{1} \frac{c}{x^2\sqrt{1-x^2}}dx\end{aligned} \tag{2.36}$$

The two separate integrals are combined to yield the final result because evaluation of the integrals as the limit approaches zero from the left and from the right cancel point-by-point. Finally, substituting $U = 1$, $L = -1$, $q = \Phi_Y(c + j\omega)$, and applying the results of (2.36) to (2.32) yields the error probability in the form of

$$P\{Y<0\} = \frac{1}{2\pi}\int_{-1}^{1}\left[\mathrm{Re}\left\{\Phi_Y\left(c + jc\frac{\sqrt{1-x^2}}{x}\right)\right\} + \frac{\sqrt{1-x^2}}{x}\mathrm{Im}\left\{\Phi_Y\left(c + jc\frac{\sqrt{1-x^2}}{x}\right)\right\}\right]\frac{dx}{\sqrt{1-x^2}} . \tag{2.37}$$

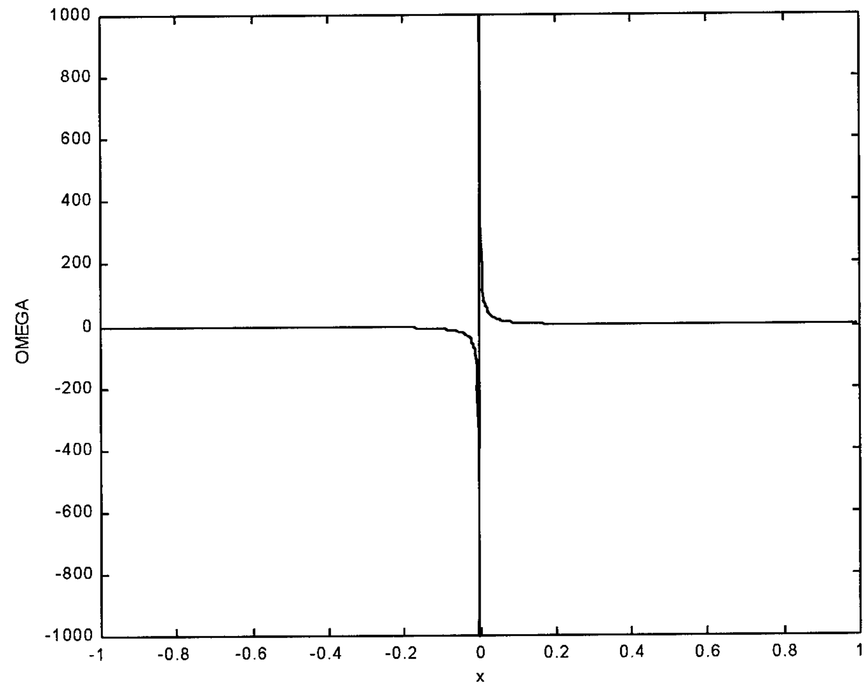

Figure 2.1. Variable Transformation $\omega = c\sqrt{1-x^2}/x$

The final step in the derivation involves applying the Gauss-Chebyshev quadrature rule defined on page 889 of [16] as

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx = \sum_{k=1}^{m} w_k f(x_k) + R_m \tag{2.38}$$

where $x_k = \cos(2k\text{-}1)\pi/2m$, $w_k = \pi/m$, and R_m is the remainder or error term. In (2.38), the integral is approximated by the summation of the weighted values of $f(x)$ evaluated at the abscissas x_k, where k = 1 to m, added to the error term [20]. Applying the Gauss-Chebyshev rule of (2.38) to the two-sided Laplace transform in the integral of (2.37) results in

$$\begin{aligned}
\Phi_Y\left(c + jc\frac{\sqrt{1-x^2}}{x}\right) &= \Phi_Y\left(c + jc\frac{\sqrt{1-\cos^2[(2k-1)\pi/2m]}}{\cos[(2k-1)\pi/2m]}\right) \\
&= \Phi_Y\left(c + jc\frac{\sqrt{\sin^2[(2k-1)\pi/2m]}}{\cos[(2k-1)\pi/2m]}\right) = \Phi_Y\left(c + jc\frac{\sin[(2k-1)\pi/2m]}{\cos[(2k-1)\pi/2m]}\right) \\
&= \Phi_Y\left(c + jc\tan[(2k-1)\pi/2m]\right) = \Phi_Y\left(c + jc\tau_k\right)
\end{aligned} \tag{2.39}$$

where $\tau_k = \tan[(2k\text{-}1)\pi/2m]$. The square root of $\sin^2(x)$ equals $\sin(x)$ as long as x is restricted to the range $(0, \pi)$. This requirement is met because of the way k and m are defined. Finally, substituting (2.39) along with $m = n/2$ and $w_i = \pi/m$ into (2.38) results in

$$\begin{aligned}
P\{Y < 0\} &= \left(\frac{1}{2\pi}\right)\left(\frac{2\pi}{n}\right)\left\{\sum_{k=1}^{n/2}\mathrm{Re}\left[\Phi_Y\left(c + jc\tau_k\right)\right]\right. \\
&\quad \left. + \frac{\sqrt{1-\cos^2[(2k-1)/2n]}}{\cos[(2k-1)/2n]}\mathrm{Im}\left[\Phi_Y\left(c + jc\tau_k\right)\right]\right\} + E_n \\
&= \frac{1}{n}\left\{\sum_{k=1}^{n/2}\mathrm{Re}\left[\Phi_Y\left(c + jc\tau_k\right)\right] + \tau_k\,\mathrm{Im}\left[\Phi_Y\left(c + jc\tau_k\right)\right]\right\} + E_n
\end{aligned} \tag{2.40}$$

where E_n is the error term. Also, n is the number of nodes and is assumed to be even.

A disparity between the development above and the development in [2] concerns the τ_k term. In the final result of [2], τ_k has $2n$ in the denominator, while in the development here there is an n term in the denominator. The derivation given here appears to be correct since, as n approaches infinity, the term τ_k approaches the function

$$\frac{\sqrt{1-x^2}}{x}. \tag{2.41}$$

This is evidenced in Figures 2.2 and 2.3. Note that, as the number of nodes increases, the correct τ_k approaches the form of the function shown in (2.41).

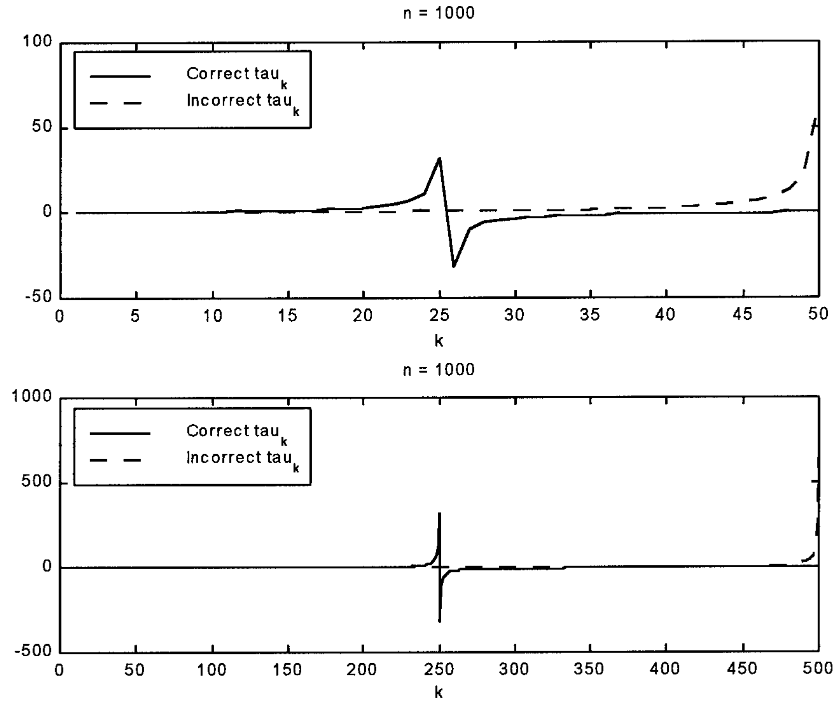

Figure 2.2 Characteristics of τ_k

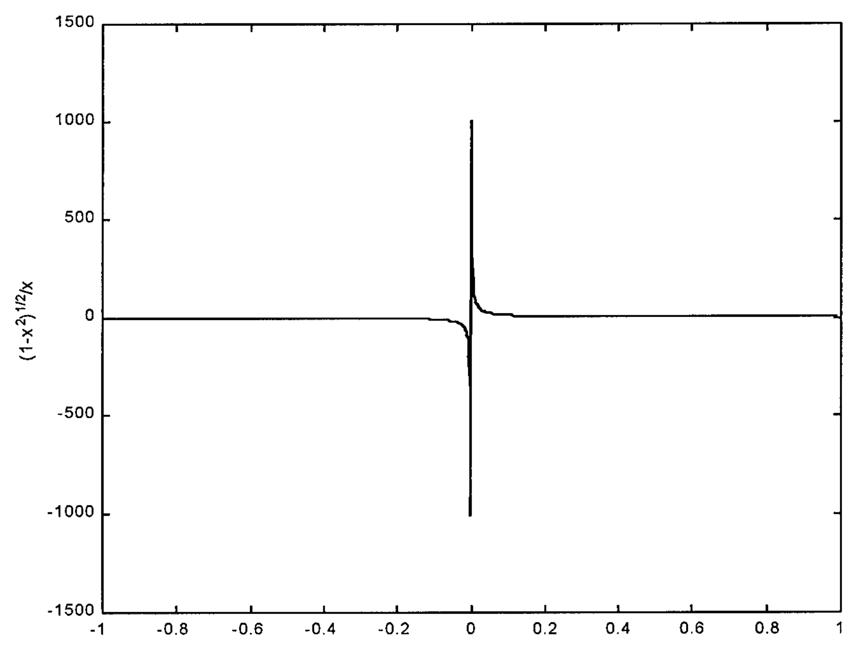

Figure 2.3 Plot of $\sqrt{1-x^2}/x$

The error term of (2.40), E_n, decreases as the number of nodes gets larger and should disappear when $n = \infty$. The variable c determines the number of nodes required to obtain a desired degree of accuracy. It is stated in [2] that through applying the method, a good value of c is that value which minimizes $\Phi_Y(c)$. Another way of choosing c, as stated in [3], is half the value of the positive singularity of $\Phi_Y(s)$ closest to the imaginary axis. One of the major objectives of this thesis is to determine exactly what makes these choices of c good. Basically, a good choice of c would enable accurate numerical results with the fewest nodes possible. This, in turn, would result in less computational time. With the method suitably defined, we proceed to test it on several communication models.

CHAPTER III

ERROR PROBABILITY ANALYSIS IN COMMUNICATION SYSTEMS USING THE GAUSS-QUADRATURE METHOD

The method presented in Chapter II is now used to analyze the error probability in several communication models of varying complexity. This analysis is accomplished to verify the utility of the method for different models. Antipodal baseband signaling in additive white Gaussian noise (AWGN), BPSK in AWGN with phase error, BPSK in AWGN and Rayleigh fading, and non-coherent frequency-shift keying (NFSK) in AWGN and Rayleigh fading are all considered in this chapter.

3.1 Antipodal Baseband Signaling in AWGN

3.1.1. Introduction. The first model considered is the simple case of antipodal baseband signaling in AWGN. A description of the model is first presented followed by a derivation of the theoretical error probability. Next, the two-sided Laplace transform is found along with the error probability using (2.40) and the results are compared to the theoretical error probability.

3.1.2. System Description. The antipodal baseband signaling set consists of the two signals

$$\left.\begin{aligned} s_1(t) &= +A \\ s_2(t) &= -A \end{aligned}\right\} \quad 0 \le t \le T_b . \tag{3.1}$$

Essentially, $s_1(t)$ is sent for the bit of 1 and $s_2(t)$ is sent for the bit of 0. As shown, each signal lasts for a duration of T_b seconds. Here, it is assumed that the signals occur with equal probability although they need not. The transmission channel is influenced by AWGN that has zero mean and a two-sided power spectral density (PSD) of $N_0/2$. The receiver used to detect the transmitted signal is depicted in Figure 3.1.

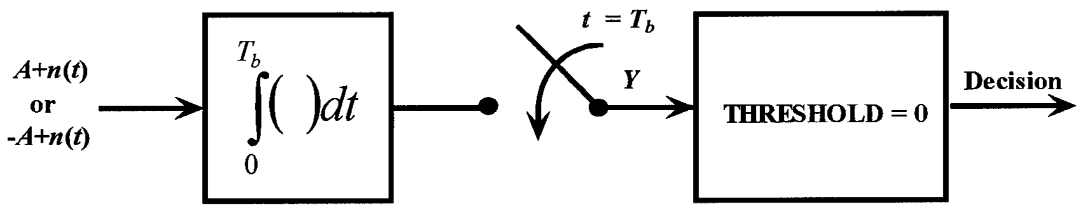

Figure 3.1 Antipodal Baseband Signaling Receiver

The output of the integrator, Y, is

$$Y = \int_0^{T_b}(\pm A + n(t))dt = \int_0^{T_b}(\pm A)dt + \underbrace{\int_0^{T_b} n(t)dt}_{N} = \pm AT_b + N. \tag{3.2}$$

For this system, the random variable input to the threshold detector has a Gaussian distribution because the additive noise is assumed to be Gaussian. In general, the probability density function (PDF) of a Gaussian random variable, Z, with mean m_Z and variance σ_Z^2 is

$$p_Z(z) = \frac{1}{\sqrt{2\pi\sigma_Z^2}}\exp\left[\frac{-(z-m_Z)^2}{2\sigma_Z^2}\right]. \tag{3.3}$$

Since $n(t)$ is Gaussian with zero mean and PSD $N_0/2$, the mean, m_N, and variance, σ_N^2, of the noise component, N, from (3.2), regardless of the signal sent, are

$$m_N = E\{N\} = E\left\{\int_0^{T_b} n(t)dt\right\} = \int_0^{T_b} \underbrace{E\{n(t)\}}_{=0} dt = 0. \tag{3.4}$$

$$\begin{aligned}\sigma_N^2 &= E\{N^2\} - \underbrace{E^2\{N\}}_{=0} = E\left\{\int_0^{T_b}\int_0^{T_b} n(t)n(t')dt'dt\right\} \\ &= \int_0^{T_b}\int_0^{T_b} \underbrace{E\{n(t)n(t')\}}_{=\frac{N_0}{2}\delta(t-t')} dt'dt = \frac{N_0}{2}\int_0^{T_b} dt = \frac{N_0 T_b}{2}\end{aligned}. \tag{3.5}$$

The random variable, Y, is conditionally Gaussian with mean and variance, conditioned on the signal sent, of

$$m_{Y|\pm A} = E\{Y|\pm A\} = E\{\pm AT_b + N\} = \underbrace{E\{\pm AT_b\}}_{=\pm AT_b} + \underbrace{E\{N\}}_{=0} = \pm AT_b \tag{3.6}$$

$$\begin{aligned}\sigma_{Y|\pm A}^2 &= E\{Y^2|\pm A\} - E^2\{Y|\pm A\} \\ &= E\{A^2T_b^2 \pm 2AT_bN + N^2\} - A^2T_b^2 \\ &= \underbrace{E\{A^2T_b^2\}}_{=A^2T_b^2} \pm \{2AT_bN\} + \underbrace{E\{N^2\}}_{N_0T_b/2} - A^2T_b^2 \\ &= 2AT_b\underbrace{E\{N\}}_{=0} + \frac{N_0T_b}{2} = \frac{N_0T_b}{2}\end{aligned}. \tag{3.7}$$

The probability of error can now be found since the PDFs of Y and N are fully described.

3.1.3 Probability of Error. At the receiver, a decision that $+A$ was sent is made if the sampled integrator output at $t = T_b$ is greater than 0, and the decision that $-A$ was sent occurs if the integrator output is less than 0. With the probability of each bit occurring half of the time, the total probability of error, P_E, is

$$P_E = \frac{1}{2}P(Y < 0 \mid +A\,\text{sent}) + \frac{1}{2}P(Y > 0 \mid -A\,\text{sent}). \tag{3.8}$$

First, the probability of error given that $+A$ was sent is found to be

$$\begin{aligned} P(Y<0\,|\,+A) &= P\left((AT_b+N)<0\,|\,+A \text{ sent}\right) = P(N<-AT_b) \\ &= \int_{-\infty}^{-AT_b} p_N(n)dn \overset{\text{By Symmetry}}{=} \int_{AT_b}^{\infty} p_N(n)dn \\ &= \frac{1}{\sqrt{2\pi(N_0T_b/2)}}\int_{AT_b}^{\infty} \exp\left[\frac{-n^2}{2(N_0T_b/2)}\right]dn \end{aligned} \quad . \tag{3.9}$$

Next, a change of variables of $u = n/(N_0T_b/2)^{1/2}$ is performed on (3.9) to result in an error probability of

$$\begin{aligned} P(Y<0\,|\,+A) &= \frac{1}{\sqrt{2\pi}}\int_{\sqrt{\frac{2A^2T_b}{N_0}}}^{\infty} \exp\left[\frac{-u^2}{2}\right]du = Q\left(\sqrt{\frac{2A^2T_b}{N_0}}\right) \\ &= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = 0.5\text{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) \end{aligned} \tag{3.10}$$

where the Q-function, $Q(u)$, and complementary error function, erfc(u), are

$$Q(u) = \frac{1}{\sqrt{2\pi}}\int_u^{\infty} e^{-t^2/2}dt \tag{3.11}$$

$$\text{erfc}(u) = \frac{2}{\sqrt{\pi}}\int_u^{\infty} e^{-t^2}dt\,. \tag{3.12}$$

The Q-function and complementary error function are related by

$$Q(u) = \frac{1}{2}\text{erfc}\left(u/\sqrt{2}\right). \tag{3.13}$$

Using a similar derivation, the error probability when $-A$ is sent can be found to be the same as that when $+A$ was sent. Therefore, the total probability of error is

$$P_E = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = 0.5\text{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right). \tag{3.14}$$

With a theoretical probability of error in hand, the Gauss-Quadrature rule is now applied.

3.1.4 Application of the Gauss-Chebyshev Quadrature Rule. To apply the Gauss-Quadrature rule; we need the two-sided Laplace transform of the PDF associated with the decision variable, Y, given $+A$ was sent. The two-sided Laplace transform of a Gaussian random variable with mean m_Y and variance σ_Y^2 is given by

$$\begin{aligned}\Phi_Y(s) &= \int_{-\infty}^{\infty} p_Y(y)e^{-sy}dy \\ &= \frac{1}{\sqrt{2\pi\sigma_Y^2}}\int_{-\infty}^{\infty}\exp\left[\frac{-(y-m_Y)^2}{2\sigma_Y^2}\right]\exp[-sy]dy \\ &= \frac{1}{\sqrt{2\pi\sigma_Y^2}}\int_{-\infty}^{\infty}\exp\left[\frac{-y^2-m_Y^2+2m_Y y-2\sigma_Y^2 sy}{2\sigma_Y^2}\right]dy \\ &= \frac{1}{\sqrt{2\pi\sigma_Y^2}}\int_{-\infty}^{\infty}\exp\left[-\left(\frac{y^2+\left(2m_Y-2\sigma_Y^2 s\right)y+m_Y^2}{2\sigma_Y^2}\right)\right]dy\end{aligned} \quad . \tag{3.15}$$

In [18], the following definite integral is given:

$$\int_{-\infty}^{\infty}\exp\left[-\left(ax^2+bx+c\right)\right]dx = \sqrt{\frac{\pi}{a}}\exp\left[\left(b^2-4ac\right)/4a\right] . \tag{3.16}$$

Now, applying this integral to (3.15) results in

$$\begin{aligned}\Phi_Y(s) &= \sqrt{\frac{1}{2\pi\sigma_Y^2}}\sqrt{\frac{\pi}{\left(1/2\sigma_Y^2\right)}}\exp\left[\frac{\left(\frac{4m_Y^2+4\sigma_Y^4 s^2-8m_Y\sigma_Y^2 s}{4\sigma_Y^4}-\frac{4m_Y^2}{4\sigma_Y^4}\right)}{4\left(\frac{1}{2\sigma_Y^2}\right)}\right] \\ &= \exp\left[\frac{4\sigma_Y^4 s^2-8m_Y\sigma_Y^2 s}{4\sigma_Y^4}\frac{2\sigma_Y^2}{4}\right] = \exp\left[\frac{0.5\sigma_Y^4 s^2-m_Y\sigma_Y^2 s}{\sigma_Y^2}\right] \\ &= \exp\left[0.5\sigma_Y^2 s^2-m_Y s\right]\end{aligned} \quad . \tag{3.17}$$

For antipodal baseband signaling, the mean and variance given that $+A$ was sent were shown in (3.6) and (3.7) to be $m_Y = AT_b$ and $\sigma_Y^2 = N_0T_b/2$, respectively. Therefore, the two-sided Laplace transform for this case is

$$\Phi_Y(s) = \exp\left(0.25N_0T_b s^2 - AT_b s\right) \tag{3.18}$$

The Gauss-Chebyshev quadrature evaluation of the error probability for antipodal baseband signaling was implemented using MATLAB and the two-sided Laplace transform given by (3.18). The results obtained are compared in Figure 3.2 with the theoretical probability of error from (3.14). As shown in Figure 3.2, the theoretical results obtained using the Gauss-Quadrature method closely match those computed using the closed form. Furthermore, only 50 nodes were used to obtain results with a maximum percent difference of 0.13%. Finally, the code used to generate these results is given in Appendix A.

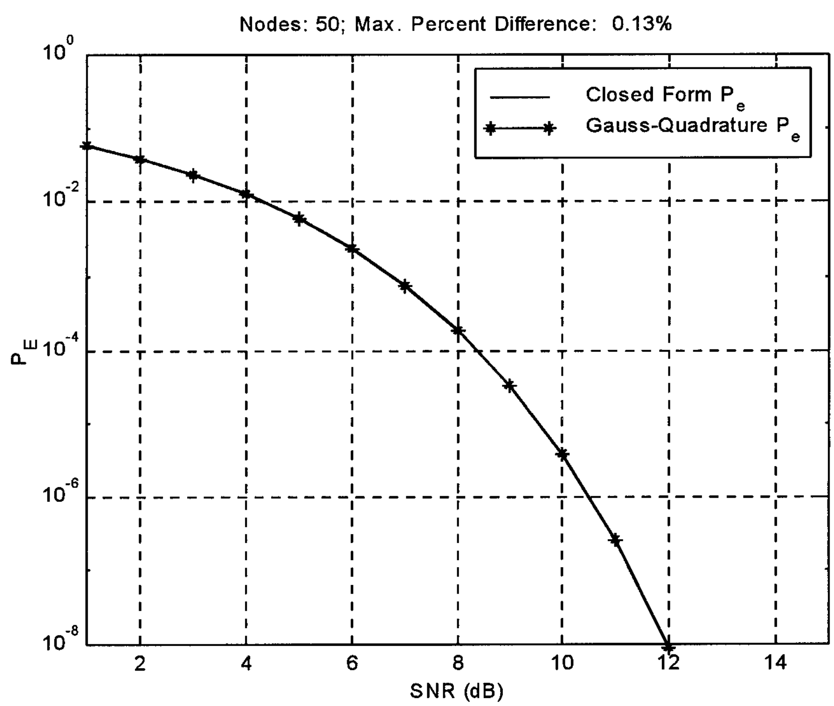

Figure 3.2 Error Probability for Antipodal Baseband Signaling

3.2 BPSK in AWGN and Imperfect Phase Reference

3.2.1 Introduction. This section considers BPSK in AWGN and an imperfect carrier reference in the receiver. The phase error resulting from the imperfect carrier reference is assumed to have a Tikinov distribution [17]. Once again, the system model is presented followed by a derivation of the error probability. The Gauss-Chebyshev quadrature rule is then applied to analyze error performance.

3.2.2 System Description. The signal set for the BPSK model is

$$\left.\begin{aligned} s_1(t) &= -A_c \cos(\omega_c t) \\ s_2(t) &= \ \ A_c \cos(\omega_c t) \end{aligned}\right\} \quad 0 \le t \le T_b . \tag{3.19}$$

Again, it is assumed that the two symbols have an equal probability of occurrence and the AWGN has a mean of zero and two-sided PSD $N_0/2$. Additionally, a coherent reference is required at the receiver and it is assumed that the phase tracking loop is imperfect; therefore, there is a phase error, ϕ, in the local reference. The receiver is depicted in Figure 3.3.

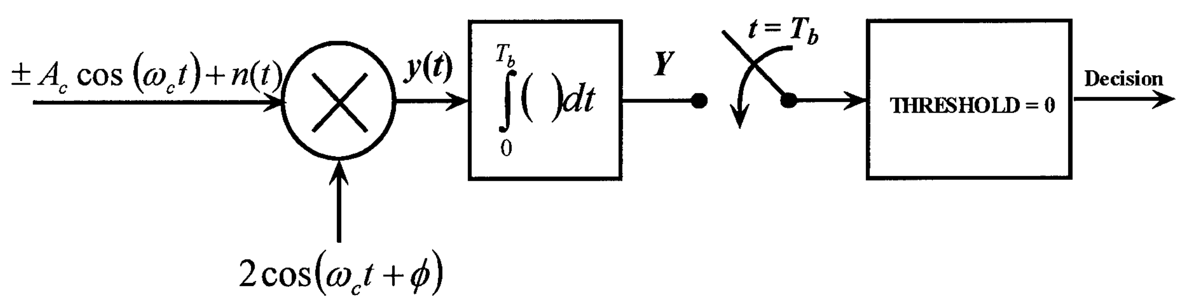

Figure 3.3 BPSK Receiver with Phase Error ϕ

The output of the integrator of Figure 3.3 is

$$
\begin{aligned}
Y &= \int_0^{T_b} 2\cos(\omega_c t+\phi)(\pm A_c \cos\omega_c t + n(t))dt \\
&= \int_0^{T_b} 2\cos(\omega_c t+\phi)(\pm A_c \cos\omega_c t)dt + \underbrace{\int_0^{T_b} 2n(t)\cos(\omega_c t+\phi)dt}_{N} \\
&= \int_0^{T_b}\left[\underbrace{\pm A_c\cos(2\omega_c t+\phi)}_{=0} \pm A_c\cos\phi\right]dt + N = \pm A_c T_b \cos\phi + N
\end{aligned}
\quad . \tag{3.20}
$$

Assuming ϕ is known, Y is again a conditionally Gaussian random variable. The mean and variance of the Gaussian random variable N, given ϕ, are found to be

$$
m_{N|\phi} = E\{N\} = E\left\{\int_0^{T_b} 2\cos(\omega_c t+\phi)n(t)dt\right\} = \int_0^{T_b} 2\cos(\omega_c t+\phi)\underbrace{E\{n(t)\}}_{=0}dt = 0 \tag{3.21}
$$

and

$$
\begin{aligned}
\sigma^2_{N|\phi} &= E\{N^2\} - \underbrace{E^2\{N\}}_{=0} \\
&= E\left\{\int_0^{T_b}\int_0^{T_b} 4\cos(\omega_c t+\phi)\cos(\omega_c t'+\phi)n(t)n(t')dt'dt\right\} \\
&= \int_0^{T_b}\int_0^{T_b} 4\cos(\omega_c t+\phi)\cos(\omega_c t'+\phi)\underbrace{E\{n(t)n(t')\}}_{=\frac{N_0}{2}\delta(t-t')}dt'dt \\
&= \frac{4N_0}{2}\int_0^{T_b}\cos^2(\omega_c t+\phi)dt \\
&= \frac{4N_0}{2}\int_0^{T_b}\left[0.5 + 0.\underbrace{5\cos(2\omega_c t+2\phi)}_{=0}\right]dt = \frac{4N_0 T_b}{4} = N_0 T_b
\end{aligned}
\quad . \tag{3.22}
$$

For simplicity, let -1 represent $s_1(t)$ and $+1$ represent $s_2(t)$. With the parameters of N known, the mean and variance of Y, given ϕ, are

$$
\begin{aligned}
m_{Y|\phi,\pm1} &= E\{Y\} = E\{\pm A_c T_b\cos\phi + N\} \\
&= E\{\pm A_c T_b \cos\phi\} + \underbrace{E\{N\}}_{=0} = \pm A_c T_b\cos\phi
\end{aligned}
\tag{3.23}
$$

and

$$\sigma^2_{Y|\phi,\pm 1} = E\left\{(Y - E(Y))^2\right\} = E\left\{(\pm A_c T_b \cos\phi + N \mp A_c T_b \cos\phi)^2\right.$$
$$= E\left\{N^2\right\} = N_0 T_b \quad . \tag{3.24}$$

3.2.3 Probability of Error. It is again desired to find the total probability of error as in (3.8). For this case, the phase error, ϕ, is assumed to be a given. The conditional error probability is

$$P_{E|\Phi=\phi} = \frac{1}{2} P(Y > 0 \mid s_1(t), \Phi = \phi) + \frac{1}{2} P(Y < 0 \mid s_2(t), \Phi = \phi). \tag{3.25}$$

Assuming that the signal $s_1(t)$ was sent, the probability that $s_2(t)$ is received is found to be

$$P(Y > 0 \mid s_1(t), \Phi = \phi) = P(N > A_c T_b \cos\phi) = \frac{1}{\sqrt{2\pi\sigma_N^2}} \int_{A_c T_b \cos\phi}^{\infty} \exp\left[-\frac{n^2}{2\sigma_N^2}\right] dn$$
$$= \frac{1}{2}\frac{2}{\sqrt{\pi}} \int_{\frac{A_c T_b \cos\phi}{\sqrt{2\sigma_N^2}}}^{\infty} \exp\left[-u^2\right] du = \frac{1}{2}\operatorname{erfc}\left[\frac{A_c T_b \cos\phi}{\sqrt{2\sigma_N^2}}\right]$$
$$= \frac{1}{2}\operatorname{erfc}\left[\sqrt{\frac{A_c^2 T_b^2}{2N_0 T_b}}\cos\phi\right] = \frac{1}{2}\operatorname{erfc}\left[\sqrt{\frac{A_c^2 T_b^2}{2N_0}}\cos\phi\right] \quad . \tag{3.26}$$
$$= \frac{1}{2}\operatorname{erfc}\left[\sqrt{\frac{E_b}{N_0}}\cos\phi\right]$$

The substitution $A_c^2 T_b/2 = E_b$ was made in (3.26). The probability of error when $s_2(t)$ is sent can be shown to be the same as that of (3.26). Therefore, the total probability of error conditioned on ϕ is

$$P_{E|\Phi=\phi} = 0.5\operatorname{erfc}\left[\sqrt{\frac{E_b}{N_0}}\cos\phi\right] = Q\left[\sqrt{\frac{2E_b}{N_0}}\cos\phi\right]. \tag{3.27}$$

The unconditional error probability is found according to [15] by multiplying the conditional error probability by the marginal PDF of Φ and integrating over ϕ:

$$P_E = \int_{-\infty}^{\infty} P_{E|\Phi=\phi} p_\Phi(\phi) d\phi . \tag{3.28}$$

The phase error is assumed to have a Tikinov PDF defined in [17] as

$$p_\Phi(\phi) = \frac{\exp(\rho\cos\phi)}{2\pi I_0(\rho)} \qquad |\phi| \le \pi \tag{3.29}$$

where $I_0(\rho)$ is the modified Bessel function of order zero. The variable ρ is the tracking loop SNR defined to be equal to the inverse of the phase error variance, σ_ϕ^2. Finally, the total probability of error is

$$\begin{aligned} P_E &= \int_{-\infty}^{\infty} P_{E|\Phi=\phi} p_\Phi(\phi) d\phi = \int_{-\pi}^{\pi} \frac{1}{2} \operatorname{erfc}\left[\sqrt{\frac{E_b}{N_0}} \cos\phi\right] \frac{\exp(\rho\cos\phi)}{2\pi I_0(\rho)} d\phi \\ &= \frac{1}{4\pi I_0(\rho)} \int_{-\pi}^{\pi} \operatorname{erfc}\left[\sqrt{\frac{E_b}{N_0}} \cos\phi\right] \exp(\rho\cos\phi) d\phi \end{aligned} . \tag{3.30}$$

A closed form solution for (3.30) is not possible. Therefore, it must be numerically integrated.

3.2.4 Application of the Gauss-Chebyshev Quadrature Rule. It is again assumed that the phase error is a given and now the conditional two-sided Laplace transform is found. But, since the conditional PDF of Y when given ϕ is Gaussian with mean and variance given in (3.23) and (3.24) respectively, the two-sided Laplace transform is known from the antipodal case to be

$$\Phi_{Y|\Phi=\phi}(s) = E\left[e^{-sY} \middle| \Phi = \phi\right] = \mathbf{exp}\left(0.5\sigma_{Y|\phi}^2 s^2 - m_{Y|\phi} s\right). \tag{3.31}$$

To find the unconditional Laplace transform, the relationship given in [15] is utilized:

$$E\{g(\overline{X},\overline{Y})\} = E\{E[g(\overline{X},\overline{Y})|\overline{X} = x]\} = \int_{-\infty}^{\infty} E\{g(\overline{X},\overline{Y})|\overline{X} = x\} p_{\overline{X}}(x) dx . \tag{3.32}$$

Assuming $s_2(t)$ is sent and a Tikinov phase error distribution, the relationship in (3.32) is used to find that the unconditional Laplace transform is

$$\begin{aligned}\Phi_Y(s) &= E\left\{e^{-sY}\right\} = \int_{-\infty}^{\infty} E\left\{e^{-sY}\middle|\Phi=\phi\right\} p_\Phi(\phi)d\phi \\ &= \int_{-\pi}^{\pi} \exp\left[0.5\sigma_{Y|\phi}^2 s^2 - m_{Y|\phi}s\right]\frac{\exp[\rho\cos\phi]}{2\pi I_0(\rho)}d\phi \\ &= \frac{\exp\left[0.5N_0T_bs^2\right]}{2\pi I_0(\rho)}\int_{-\pi}^{\pi}\exp[\cos\phi(-A_cT_bs+\rho)]d\phi\end{aligned} \quad . \tag{3.33}$$

The following relationship is defined in [19]:

$$\int_{-\pi}^{\pi}\exp[\beta\cos\phi]d\phi = 2\pi I_0(\beta). \tag{3.34}$$

Applying this relationship to (3.33) allows the two-sided Laplace transform to be expressed in closed form as

$$\Phi_Y(s) = \frac{I_0(\rho - sA_cT_b)\exp\left[0.5N_0T_bs^2\right]}{I_0(\rho)}. \tag{3.35}$$

MATLAB is again used to calculate error probabilities. Several values of loop SNR are considered. The loop SNR is set at a desired constant level above E_b/N_0. For example, the PLL SNR is maintained at 0 dB above E_b/N_0 in the graph of Figure 3.4. The results are depicted in Figures 3.4-3.6. Time savings between numerical integration with MATLAB's *quad8* command and the method under consideration are noted at the top of each graph. Additionally, each graph also has a plot of the error probability if the phase error were zero. As shown, the results for both methods match to within 1%. More importantly, there is up to 99% computational time savings realized when using the Gauss-Chebyshev quadrature method as compared with using the *quad8* command. The code used to create these results is shown in Section 2 of Appendix A.

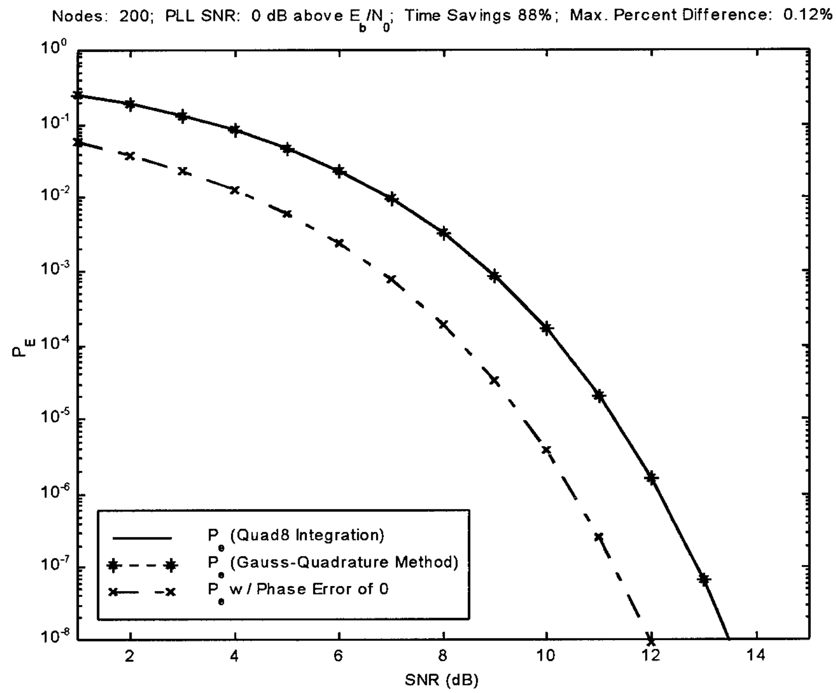

Figure 3.4 Error Probability for BPSK with Loop SNR of 0 dB Above E_b/N_0

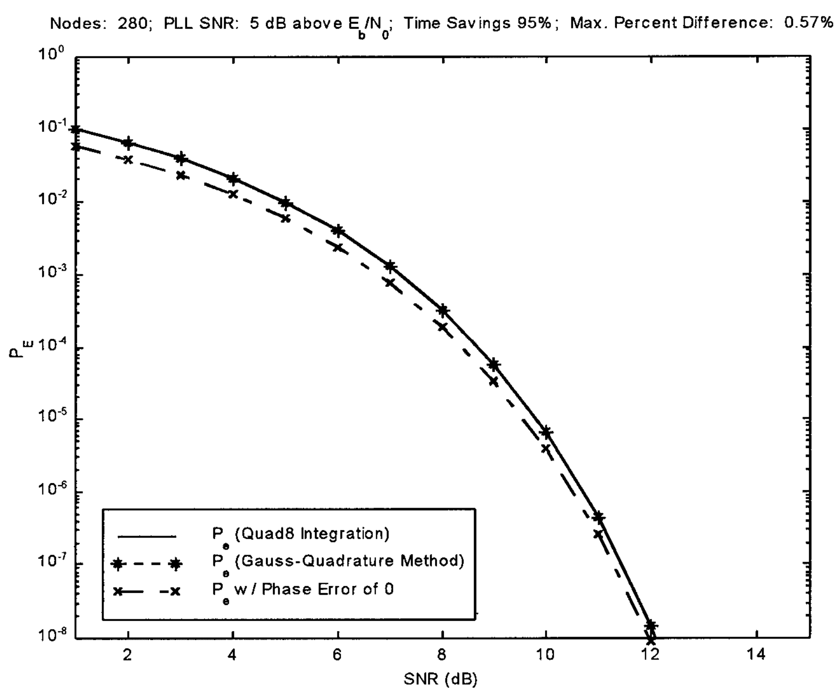

Figure 3.5 Error Probability for BPSK with Loop SNR of 5 dB Above E_b/N_0

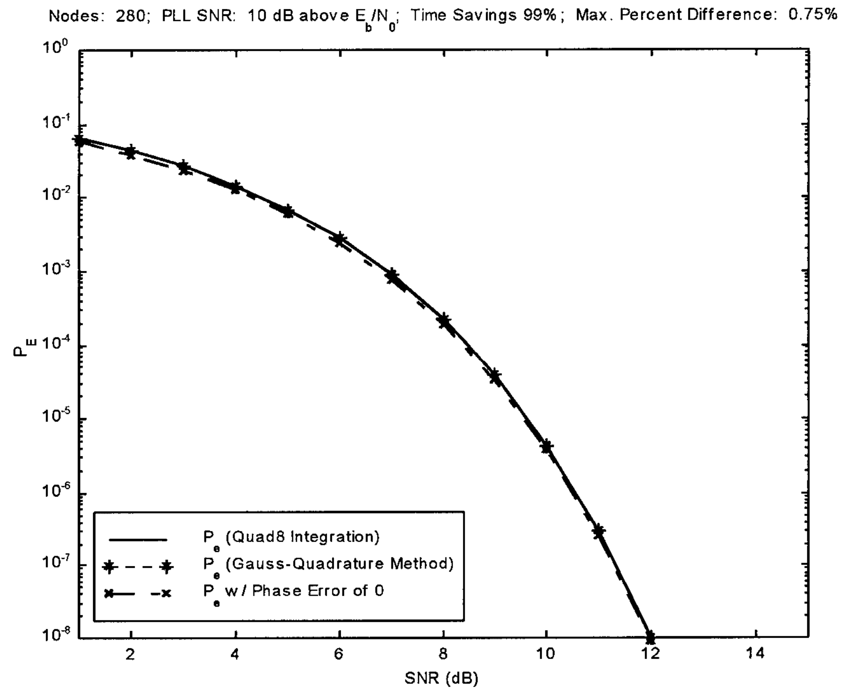

Figure 3.6 Error Probability for BPSK with Loop SNR of 10 dB Above E_b/N_0

3.3 BPSK in AWGN and Rayleigh Fading Channel

3.3.1 Introduction. The next model under consideration is BPSK in a channel influenced by AWGN and Rayleigh fading. The Rayleigh fading is assumed to be frequency non-selective and to vary slowly over a bit period. Again the system model is described, the error probability is derived, and the Gauss-Quadrature rule is applied.

3.3.2 System Description. The signal set for this model is

$$\left.\begin{aligned} s_1(t) &= -A_c \cos(\omega_c t) \\ s_2(t) &= A_c \cos(\omega_c t) \end{aligned}\right\} \quad 0 \le t \le T_b . \tag{3.36}$$

The signal set is obviously the same as that in the other BPSK case and is reprinted here for convenience. Again, it is assumed that the two symbols have equal *a priori* probability and the AWGN has a mean of zero and two-sided PSD $N_0/2$. The Rayleigh fading in the communication channel is a random, multiplicative process that causes the received signal to have a random amplitude. This process is assumed to be frequency non-selective so it affects all frequencies in the same manner. Additionally, the assumption that the Rayleigh fading is slow results in the received signal being constant over a bit interval. This allows the phase to be estimated without error [21]. Figure 3.7 shows the receiver used in this model.

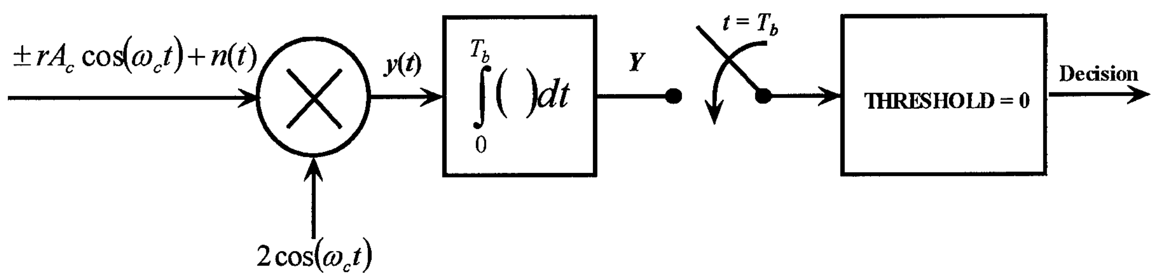

Fig. 3.7 BPSK Receiver

The variable r is the Rayleigh random variable. In a similar manner as that of the previous BPSK case, the output of the integrator is found to be

$$Y = \pm rA_cT_b + N. \tag{3.37}$$

The Gaussian random variable, N, has a mean and variance of

$$m_N = E\{N\} = E\left\{\int_0^{T_b} 2\cos(\omega_c t)n(t)dt\right\} = \int_0^{T_b} 2\cos(\omega_c t)\underbrace{E\{n(t)\}}_{=0}dt = 0 \tag{3.38}$$

and

$$\sigma_N^2 = E\{N^2\} - \underbrace{E^2\{N\}}_{=0} = \int_0^{T_b}\int_0^{T_b} 4\cos(\omega_c t)\cos(\omega_c t')\underbrace{E\{n(t)n(t')\}}_{=\frac{N_0}{2}\delta(t-t')}dt'dt$$
$$= \frac{4N_0}{2}\int_0^{T_b}\cos^2(\omega_c t)dt = N_0 T_b \quad . \tag{3.39}$$

Again, $s_1(t)$ is represented by -1 and $s_2(t)$ by $+1$. The mean and variance of Y, conditioned on r, are

$$m_{Y|R=r,\pm 1} = E\{Y\} = E\{\pm rA_c T_b + N\} = \pm rA_c T_b \tag{3.40}$$

and

$$\sigma^2_{Y|R=r,\pm 1} = E\left[(Y - E(Y))^2\right] = E\left[(\pm rA_c T_b + N \mp rA_c T_b)^2\right] = N_0 T_b \, . \tag{3.41}$$

3.3.3 Probability of Error. In a similar manner as in the previous case, the conditional probability of error is

$$P_{E|R=r} = 0.5\mathrm{erfc}\left[\sqrt{\frac{E_b}{N_0}}\right] = 0.5\mathrm{erfc}\left[\sqrt{\frac{r^2 A_c^2 T_b}{2N_0}}\right] = 0.5\mathrm{erfc}\left[\sqrt{\gamma_b}\right] \tag{3.42}$$

where $\gamma_b = E_b/N_0 = 0.5r^2A_c^{\,2}T_b/N_0$ is the SNR. To find the unconditional error probability, the product of the conditional error probability and the marginal PDF of R^2 is integrated over r^2. Since r is Rayleigh distributed, r^2 has a chi-square distribution with two degrees of freedom [21]. Similarly, γ_b also has a PDF which is chi-square with two degrees of freedom and is represented by

$$p(\gamma_b) = \frac{1}{\gamma'_b} e^{-\gamma_b/\gamma'_b}, \;\; \gamma_b \geq 0 \tag{3.43}$$

where $\gamma'_b = \gamma_b E(r^2)$ is the average SNR and $E(r^2)$ is the average value of r^2. Therefore, the unconditional error probability is

$$P_E = \int_{-\infty}^{\infty} p(\gamma_b) P_{E|\gamma_b} d\gamma_b = \frac{1}{\gamma'_b} \int_0^{\infty} e^{-\gamma_b/\gamma'_b} \left[\frac{1}{\sqrt{2\pi}} \int_{\sqrt{2\gamma_b}}^{\infty} e^{-u^2/2} du \right] d\gamma_b . \tag{3.44}$$

With a variable substitution of $z = u/(2\gamma_b)^{1/2}$ and interchanging the order of integration, the error probability is

$$P_E = \frac{1}{\sqrt{\pi\gamma'_b}} \int_1^{\infty} \left[\int_0^{\infty} \sqrt{\gamma_b} \exp\left[-\gamma_b \left(\frac{1}{\gamma'_b} + z^2 \right) \right] d\gamma_b \right] dz . \tag{3.45}$$

Applying the integral

$$\int_0^{\infty} x^n e^{-ax} dx = \frac{\Gamma(n+1)}{a^{n+1}} \tag{3.46}$$

from [18] to (3.44) yields

$$P_E = \frac{1}{\sqrt{\pi\gamma'_b}} \int_1^{\infty} \left[\frac{\Gamma(0.5+1)}{\left((1/\gamma'_b) + z^2\right)^{3/2}} \right] dz = \frac{1}{2\gamma'_b} \int_1^{\infty} \frac{1}{\left((1/\gamma'_b) + z^2\right)^{3/2}} dz . \tag{3.47}$$

Finally, applying the indefinite integral

$$\int \frac{dx}{\left(ax^2 + bx + c\right)^{3/2}} = \frac{2(ax+b)}{\left(4ac - b^2\right)\sqrt{ax^2 + bx + c}} \tag{3.48}$$

from [18] to (3.47) results in the error probability

$$P_E = \frac{1}{2\gamma'_b} \left[1 - \sqrt{\frac{\gamma'_b}{1+\gamma'_b}} \right] \tag{3.49}$$

3.3.4 Application of the Gauss-Chebyshev Quadrature Rule. The decision variable, Y, of (3.37), given the variable r and that $s_2(t)$ was sent, has a Gaussian distribution with $m_Y = rA_cT_b$ and $\sigma_Y^2 = N_0T_b$. Therefore, the conditional two-sided Laplace transform has the familiar form

$$\Phi_{Y|r}(s) = \exp\left(.5N_0T_bs^2 - rA_cT_bs\right). \tag{3.50}$$

The PDF of r is given in [21] to be

$$p_R(r) = \frac{2r}{E(r^2)} \exp\left[-\frac{r^2}{E(r^2)}\right],\ r \geq 0. \tag{3.51}$$

The unconditional Laplace transform is then

$$\Phi_Y(s) = \int_{-\infty}^{\infty} \Phi_{Y|r}(s) p_R(r) dr = \frac{2\exp\left[0.5 N_0 T_b s^2\right]}{E(r^2)} \int_0^{\infty} r \exp\left[-\frac{1}{E(r^2)} r^2 - A_c T_b s r\right] dr. \tag{3.52}$$

Applying the integral

$$\int_0^{\infty} u \exp\left[-au^2 - 2bu\right] du = \frac{1}{2a} - \frac{b}{a} \exp\left(\frac{b^2}{a}\right)\left[\frac{1}{2}\sqrt{\frac{\pi}{a}} - \frac{b}{a}\int_0^1 \exp\left(-\frac{b^2}{a}x^2\right) dx\right] \tag{3.53}$$

from Appendix B to (3.52) yields

$$\Phi_Y(s) = e^{.5 N_0 T_b s^2}\left\{1 - e^{\left(-\frac{E_b' T_b s^2}{2}\right)}\left[s\sqrt{\frac{\pi E_b' T_b}{2}} - E_b' T_b s^2 \int_0^1 \exp^{\left(-\frac{E_b' T_b s^2}{2} x^2\right)} dx\right]\right\} \tag{3.54}$$

where $E_b' = E(r^2) E_b$ is the average bit energy. The last term of (3.54) could not be found in closed form and will be computed numerically.

The resulting Laplace transform of (3.54) is now implemented in MATLAB using the Gauss-Chebyshev quadrature method to obtain the error probability. This is compared to the error probability from (3.49) in Figure 3.8. Figure 3.8 shows that less than a one percent difference is obtained with 30 nodes. However, the method took considerably longer to implement than the closed form solution due to the numerical integration required. The transform of (3.54) can be expressed in terms of a complementary error function and this would likely decrease computation time. However, the argument of the complementary error function is complex and MATLAB can not implement this. Additionally, a method to find a good value for the constant c

was successful. This will be discussed further in Chapter IV. The code again is shown in Appendix A, Section 3.

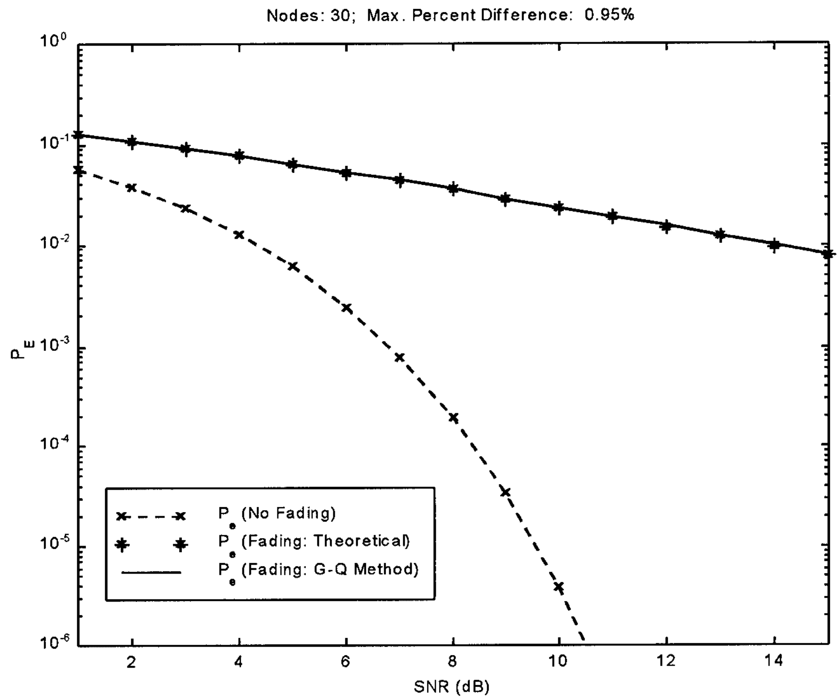

Figure 3.8. Error Probability for BPSK in AWGN and Rayleigh Fading

3.4 NFSK IN AWGN AND RAYLEIGH FADING CHANNEL

3.4.1 Introduction. The final model tested is NFSK in AWGN and Rayleigh fading. Again, the Rayleigh fading is assumed to be frequency non-selective and to vary slowly over a bit period. The system model is first described, followed by a development of the error probability and the application of the Gauss-Chebyshev quadrature rule.

3.4.2 System Description. The signal set for this NFSK case is

$$\left.\begin{aligned} s_1(t) &= A_c \cos(\omega_1 t + \theta) \\ s_2(t) &= A_c \cos(\omega_2 t + \theta) \end{aligned}\right\} \quad 0 \le t \le T_b . \tag{3.55}$$

The frequency separation between ω_1 and ω_2 is assumed to be large enough so that $s_1(t)$ and $s_2(t)$ occupy non-overlapping spectral bands [17]. Again, it is assumed that the two symbols have equal *a priori* probability and the AWGN has zero mean and variance $N_0/2$. The Rayleigh fading is again assumed to be frequency non-selective and to vary slowly over a bit interval. The receiver used for this model is shown in Figure 3.9.

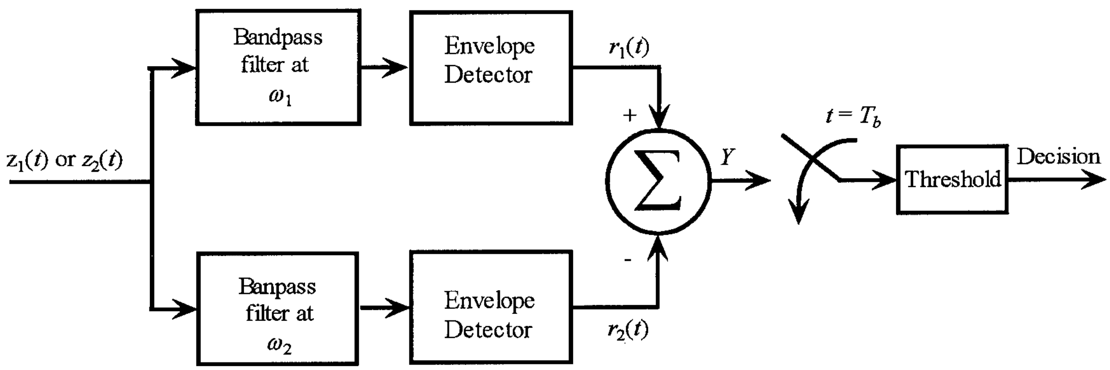

Figure 3.9. NFSK Receiver

The derivation will proceed by first considering the case where there is no Rayleigh fading in the channel and then proceed to account for the fading. The two possible received signals during a given bit interval, as defined in [17], are

$$\left.\begin{aligned} z_1(t) &= \sqrt{\frac{2E_b}{T_b}}\cos(\omega_1 t + \alpha) + n(t) \\ z_2(t) &= \sqrt{\frac{2E_b}{T_b}}\cos(\omega_2 t + \alpha) + n(t) \end{aligned}\right\} \quad 0 \le t \le T_b \tag{3.56}$$

where $E_b = 0.5A_c^2T_b$, α is the random phase factor that is uniformly distributed over $(0, 2\pi)$, and $n(t)$ is the AWGN. The received signal set is then resolved into the space defined by the orthogonal basis set

$$\phi_{x1}(t)=\sqrt{\frac{2}{T_b}}\cos\omega_1 t,\quad \phi_{x2}(t)=\sqrt{\frac{2}{T_b}}\sin\omega_1 t$$
$$\phi_{y1}(t)=\sqrt{\frac{2}{T_b}}\cos\omega_2 t,\quad \phi_{y2}(t)=\sqrt{\frac{2}{T_b}}\sin\omega_2 t \tag{3.57}$$

With the basis set in hand and assuming that $s_1(t)$ was sent, the received signal is defined by the data vector $\boldsymbol{Z} = (X_1, X_2, Y_1, Y_2)$ where the components are

$$X_1 = \sqrt{E_b}\cos\alpha + N_{x1}, X_2 = N_{x2}$$
$$Y_1 = -\sqrt{E_b}\sin\alpha + N_{y1},\ Y_2 = N_{y2}, \tag{3.58}$$

where $N_{x1,2}$ and $N_{y1,2}$ are uncorrelated noise components with zero means and variances $N_0/2$. Next, the PDFs of the random variables R_1 and R_2 defined as

$$R_1 = \sqrt{X_1^2 + Y_1^2} \tag{3.59}$$

and

$$R_2 = \sqrt{X_2^2 + Y_2^2} \tag{3.60}$$

are sought. Given the random variable α, X_1 and Y_1 are Gaussian random variables with means of $(E_b)^{1/2}\cos\alpha$ and $-(E_b)^{1/2}\sin\alpha$, respectively. The variance of both X_1 and Y_1 are $N_0/2$. Additionally, X_2 and Y_2 are also Gaussian with zero means and variances of $N_0/2$. Because X_1 and Y_1 are independent, the joint PDF of X_1 and Y_1 is found by multiplying the marginal PDFs of X_1 and Y_1 to result in

$$p_{X_1Y_1|\alpha}(x_1, y_1|\alpha) = \frac{1}{\pi N_0}\exp\left[-\frac{1}{N_0}\left[\left(x_1 - \sqrt{E_b}\cos\alpha\right)^2 + \left(y_1 + \sqrt{E_b}\sin\alpha\right)^2\right]\right]. \tag{3.61}$$

Similarly, the joint PDF of X_2 and Y_2, which does not depend on α, is

$$p_{X_2Y_2}(x_2, y_2) = \frac{1}{\pi N_0}\exp\left[-\frac{1}{N_0}\left[x_2^{\ 2} + y_2^{\ 2}\right]\right] \tag{3.62}$$

The PDFs of (3.61) and (3.62) are then transformed into the polar coordinates $r_{1,2}$ and $\phi_{1,2}$ defined as

$$\left.\begin{aligned} x_{1,2} &= r_{1,2}\sqrt{\frac{N_0}{2}}\sin\phi_{1,2} \\ y_{1,2} &= r_{1,2}\sqrt{\frac{N_0}{2}}\cos\phi_{1,2} \end{aligned}\right\} \qquad r_{1,2} > 0;\ 0 < \phi_{1,2} \le 2\pi \tag{3.63}$$

In general, a variable transformation from the random variables U and V into $Z = g(U,V)$ and $W = h(U,V)$ results in a joint PDF in terms of Z and W defined in [15] as

$$p_{ZW}(z,w) = \frac{p_{U,V}(u_1,v_1)}{|J(u_1,v_1)|} + \cdots \frac{p_{U,V}(u_n,v_n)}{|J(u_n,v_n)|} + \cdots. \tag{3.64}$$

The terms u_n and v_n represent the real roots of the transformation equations and $J(u,v)$ is the Jacobian of the transformation defined as

$$J(u,v) = \begin{vmatrix} \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \\ \dfrac{\partial w}{\partial u} & \dfrac{\partial w}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial u}{\partial z} & \dfrac{\partial u}{\partial w} \\ \dfrac{\partial v}{\partial z} & \dfrac{\partial v}{\partial w} \end{vmatrix}^{-1}. \tag{3.65}$$

In this transformation, the Jacobian is

$$J(u,v) = \begin{vmatrix} \sqrt{\dfrac{N_0}{2}}\sin\phi & \sqrt{\dfrac{N_0}{2}}r\cos\phi \\ \sqrt{\dfrac{N_0}{2}}\cos\phi & -\sqrt{\dfrac{N_0}{2}}r\sin\phi \end{vmatrix}^{-1} = -\frac{2}{rN_0}. \tag{3.66}$$

Applying the transformation to (3.61) and averaging over the random variable α results in

$$p_{R_1\Phi_1}(r_1,\phi_1) = \frac{r_1}{2\pi}\exp\left[-\frac{1}{2}\left(r_1^2 + \frac{2E_b}{N_0}\right)\right]I_0\left(r_1\sqrt{\frac{2E_b}{N_0}}\right) \quad r_1 > 0,\ 0 < \phi_1 \le 2\pi. \tag{3.67}$$

Additionally, the joint PDF of R_2 and Φ_2 is

$$p_{R_2\Phi_2}(r_2,\phi_2) = \frac{r_2}{2\pi}\exp\left[-\frac{r_2^2}{2}\right] \quad r_2 > 0,\ 0 < \phi_2 \le 2\pi . \tag{3.68}$$

Finally, integrating (3.67) and (3.68) over $\Phi_{1,2}$, respectively, results in

$$p_{R_1}(r_1) = r_1 \exp\left[-\frac{1}{2}\left(r_1^2 + \frac{2E_b}{N_0}\right)\right] I_0\left(r_1\sqrt{\frac{2E_b}{N_0}}\right) \quad r_1 > 0 \tag{3.69}$$

and

$$p_{R_2}(r_2) = r_2 \exp\left[-\frac{r_2^2}{2}\right] \quad r_2 > 0 . \tag{3.70}$$

From (3.69) and (3.70), R_1 is Rician distributed and R_2 is Rayleigh distributed. A similar derivation in the case where $s_2(t)$ was sent results in a Rician distribution for R_2 and a Rayleigh distribution for R_1. Now, with the unconditional probabilities of R_1 and R_2 in hand, the error probability is found.

3.4.3 Probability of Error. The decision about which signal was transmitted is based upon the random variables R_1 and R_2. Since is was assumed that $s_1(t)$ was transmitted, an error occurs if $R_1 < R_2$. The probability that R_1 is less than R_2 is

$$\begin{aligned} P(R_1 < R_2 | R_1) &= \int_{R_1}^{\infty} p_{R_2}(r_2)dr_2 = \int_{R_1}^{\infty} r_2 \exp\left(-\frac{r_2^2}{2}\right)dr_2 \\ &= \int_{R_1^2/2}^{\infty} \exp(-u)du = \exp\left(-\frac{R_1^2}{2}\right) \end{aligned} \tag{3.71}$$

where a variable substitution of $u = r_2^2/2$ was made. The probability in (3.71) is then averaged over R_1 to obtain a probability of error of

$$P_E = \int_{-\infty}^{\infty} P(R_1 < R_2 | R_1) p_{R_1}(r_1) dr_1 = e^{-E_b/N_0} \int_0^{\infty} r_1 \exp(-r_1^2) I_0\left(r_1\sqrt{\frac{2E_b}{N_0}}\right) dr_1 . \tag{3.72}$$

Applying the integral

$$\int_0^\infty xe^{-ax^2} I_0(bx)dx = \frac{1}{2a}e^{b^2/4a} \tag{3.73}$$

from [17] to (3.72) obtains the closed-form error probability

$$P_E = \frac{1}{2}e^{-E_b/2N_0}. \tag{3.74}$$

A similar derivation in the case where $s_2(t)$ is sent results in the same error probability as in (3.74); therefore, (3.74) is the total error probability.

The next step in the derivation is to consider the introduction of Rayleigh fading to the channel. The inclusion of this random variable, g, to the model results in a received signal of the form

$$z_{1,2}(t) = \sqrt{\frac{2g^2E_b}{T_b}}\cos(\omega_{1,2}t + \alpha) + n(t) \quad 0 \le t \le T_b. \tag{3.75}$$

Since g is a Rayleigh random variable, g^2 will have a chi-square distribution with two-degrees of freedom. Consequently, the signal-to-noise ratio,

$$\gamma_b = \frac{E(g^2)E_b}{N_0} \tag{3.76}$$

will also have a chi-square distribution with two-degrees of freedom of the form

$$p(\gamma_b) = \frac{1}{\gamma'_b}e^{-\gamma_b/\gamma'_b},\ \gamma_b \ge 0 \tag{3.77}$$

where $\gamma'_b = \gamma_b E(g^2)$ is again the average signal-to-noise ratio as in the BPSK in Rayleigh fading case. Finally, averaging over γ_b results in a total unconditional probability of error of

$$P_E = \int_{-\infty}^{\infty} p(\gamma_b) P_{E|\gamma_b} d\gamma_b = \frac{1}{2\gamma'_b} \int_0^{\infty} \exp\left[-\gamma_b\left(\frac{1}{2}+\frac{1}{\gamma'_b}\right)\right] d\gamma_b$$
$$= \frac{1}{2\gamma'_b} \frac{1}{\left(\frac{1}{2}+\frac{1}{\gamma'_b}\right)} = \frac{1}{2+\gamma'_b} \quad . \tag{3.78}$$

3.4.4 Application of the Gauss-Chebyshev Quadrature Rule. The final step for this case is to again find the two-sided Laplace transform of a decision metric. First, the transform is found in the absence of Rayleigh fading. In this case, the decision metric, assuming that R_1 is associated with the signal sent, is

$$Y = R_1^2 - R_2^2 . \tag{3.79}$$

As shown in (3.79), an error occurs if $Y < 0$, which corresponds to $R_2{}^2 > R_1{}^2$. Making use of the PDFs of R_1 and R_2 from (3.69) and (3.70), the two-sided Laplace transform is

$$\Phi_Y(s) = E\left\{e^{-sY}\right\} = E\left\{e^{-sR_1^2} e^{sR_2^2}\right\}$$
$$= \int_0^{\infty}\left[\int_0^{\infty} r_1 \exp\left[-r_1^2(0.5+s)\right] I_0\left(r_1\sqrt{\gamma_b}\right) dr_1\right] r_2 \exp\left[-r_2^2(0.5-s)\right] dr_2 \tag{3.80}$$

where the integration could be performed separately because R_1 and R_2 are independent random variables. Applying the integral of (3.73) to the inner integral of (3.80) results in

$$\Phi_Y(s) = \frac{\exp\left[-\gamma_b\left(1-\frac{1}{2(0.5+s)}\right)\right]}{2(0.5+s)} \int_0^{\infty} r_2 \exp\left[-r_2^2(0.5-s)\right] dr_2 \, . \tag{3.81}$$

Next, applying the integral from [18]

$$\int_0^{\infty} x^m e^{-ax^2} dx = \frac{\Gamma[(m+1)/2]}{2a^{(m+1)/2}}, \tag{3.82}$$

where

$$\Gamma(n+1) = n!, \tag{3.83}$$

to (3.81) results in

$$\Phi_Y(s)=\frac{\exp\left[-\gamma_b\left(1-\frac{1}{2(0.5+s)}\right)\right]}{4(0.5+s)(0.5-s)}. \tag{3.84}$$

Now consider the case where the Rayleigh random variable g again affects the amplitude and the signal-to-noise ratio. Therefore, the result in (3.84) is conditioned on the random SNR γ_b. To obtain the unconditional Laplace transform requires integrating the product of the conditional transform with the PDF of γ_b to yield

$$\begin{aligned}\Phi_Y(s)&=\int_{-\infty}^{\infty}\Phi_{Y|\gamma_b}(s)p(\gamma_b)d\gamma_b=\frac{1}{\gamma'_b}\int_0^{\infty}\frac{\exp\left[-\gamma_b\left(\frac{1}{\gamma'_b}+1-\frac{1}{2(0.5+s)}\right)\right]}{4(0.5+s)(0.5-s)}d\gamma_b\\&=\frac{1}{4\gamma'_b(0.5+s)(0.5-s)\left(\frac{1}{\gamma'_b}+1-\frac{1}{2(0.5+s)}\right)}\end{aligned} \tag{3.85}$$

The error probability, using (3.85) and the Gauss-Chebyshev quadrature method, is again implemented using MATLAB and the result compared to those obtained from (3.78). As described in Chapter II, the choice for the constant c was set to half of the smallest positive singularity. Additionally, the error probability for the case where the channel is only influenced by AWGN, from (3.74), is also shown for reference. The results are depicted in Figure 3.10. As shown in Figure 3.10, only ten nodes are required to obtain a maximum percent difference of less than one percent between the two methods. The fact that few nodes are required gives comparable computation time between the two methods. The code is shown in Section 4 of Appendix A.

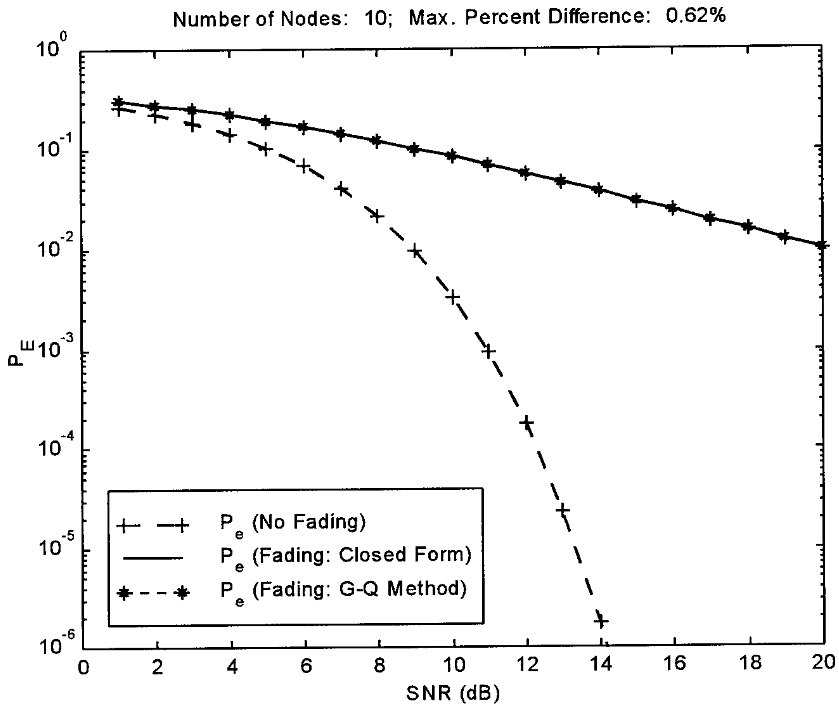

Figure 3.10. Error Probability for NFSK in AWGN and Rayleigh Fading

CHAPTER IV

DISCUSSION OF THE METHOD

This chapter discusses some of the observations made when using the Gauss-Chebyshev quadrature method defined in Chapter II and applied in Chapter III. A section discussing some general observations is first covered followed by more detail on how to choose the constant c.

4.1 General Observations

The utility of this method to carry out error probability analysis was shown in Chapter III. As shown, the implementation involves defining a suitable decision metric and then finding the two-sided Laplace transform of the PDF associated with that metric. The method used to obtain this transform varied slightly between the different models considered. The antipodal baseband signaling case involved a fairly straightforward derivation of the Laplace transform. Using the result from this antipodal case in the BPSK in AWGN with an imperfect carrier reference allowed us to assume a Gaussian distribution for the decision metric. Furthermore, this allowed the use of the already known form for the Laplace transform of a Gaussian PDF. Then, it was a relatively easy step to find a closed form Laplace transform by integrating the product of the conditional transform with the phase error distribution. Therefore, it is seen how the second case

built on the first. A similar derivation was used in the BPSK in AWGN with Rayleigh fading case but a closed form solution was not obtained.

Applying the technique to the NFSK case where the channel was influenced by AWGN and Rayleigh fading required a somewhat different approach. A closed form solution for the Laplace transform was found in the NFSK case. The point is that there is no single way of easily finding the Laplace transform of the PDF associated with the decision metric. If one approach toward this end is not successful in obtaining a closed form transform, then another approach may be in order. A closed form solution is obviously desired in order to reduce the computational complexity when implementing the Gauss-Chebyshev quadrature numerical technique. Additionally, the utility of using results from one case to solve for transform of another case shows how the method can save time. The most important factor in obtaining accurate results for this method revolves around finding a good choice for the constant c, which is now discussed.

4.2 Choosing the Constant *c*

The most important factor in implementing this method was setting the constant c. As stated in the method development, a proper choice for c enables accurate results to be obtained for the error probability while using the fewest nodes possible. This obviously equates to requiring less computational time. To recap, it was stated in [2] that a good choice for c is that value of c which minimizes $\Phi_Y(c)$. This corresponds to the Chernoff bound. Another way of choosing c, as stated in [3], is by setting it to half the value of the positive singularity of $\Phi_Y(s)$ which is closest to the imaginary axis. In most cases, setting

the value of c according to either of these two methods will result in accurate results with minimal nodes.

For example, consider the NFSK in AWGN and Rayleigh fading case. The two-sided Laplace transform for this model is

$$\Phi_Y(s) = \frac{1}{4\gamma'_b(0.5+s)(0.5-s)\left(\frac{1}{\gamma'_b} + 1 - \frac{1}{2(0.5+s)}\right)} \tag{4.1}$$

from (3.85), repeated here for illustration. As shown in (4.1), the positive singularity at $s = 0.5$ is that which is positive and closest to the imaginary axis. Therefore, the value for c in this case was set to 0.25. Indeed, this resulted in good performance, giving a maximum percent difference of 0.62%, as shown in Figure 3.10. The function $\Phi_Y(c)$ for this case is plotted in Figure 5.1 for several values of SNR. This value of c worked so well because at $c = 0.25$ the function $\Phi_Y(c)$ is well behaved, as shown in Figure 4.1. The integral of $\Phi_Y(s)$ is thus fairly easy to approximate using the Gauss-Chebyshev quadrature rule of (2.38). This is obvious because a straight line is much easier to approximate than say a quadratic. For comparison, at $c = 0.4$, $\Phi_Y(c)$ is less well behaved and a maximum percentage difference of 6.7% results when using this value of c over the same SNR range as when $c = 0.25$.

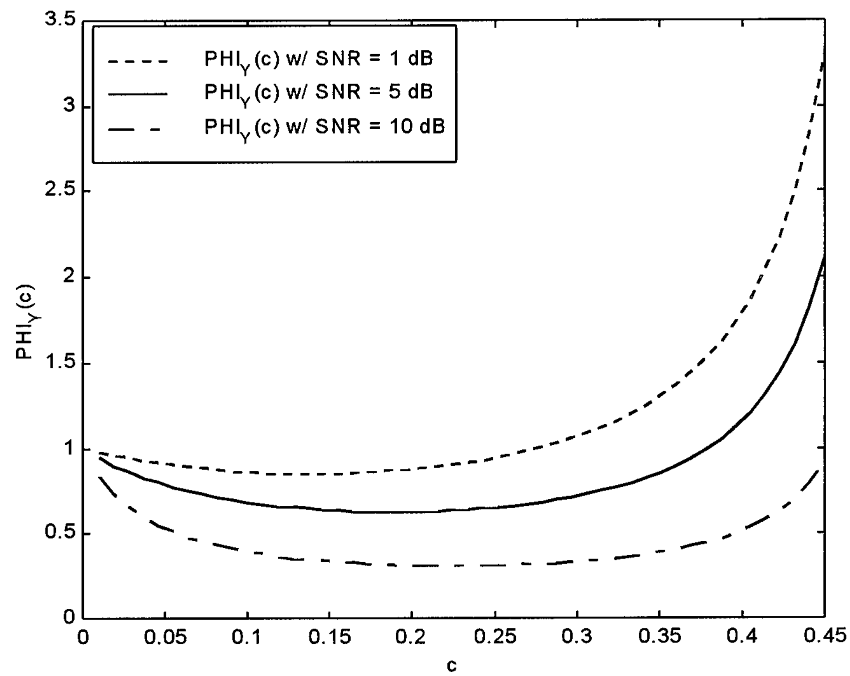

Figure 4.1. $\Phi_Y(c)$ for NFSK in AWGN and Rayleigh Fading

With the NFSK example in hand, it can be stated that a good value for c is that value for which $\Phi_Y(c)$ is well behaved. Essentially, this corresponds to where the slope of $\Phi_Y(c)$ is unchanging or, in other words, where the second derivative of $\Phi_Y(c)$ is zero. This hypothesis is tested with the NFSK case because it is evident from Figure 4.1 that $c = 0.25$ may not be a good choice for all values of SNR. Therefore, a technique is implemented to obtain values of c where the second derivative of $\Phi_Y(c)$, at the different SNR values, is zero. These c values are then used in implementing the Gauss-Chebyshev quadrature method. There was no appreciable difference between results obtained when c is set to 0.25 versus c chosen as described here. However, when the c optimization technique is applied to the BPSK in AWGN and Rayleigh fading case, performance is improved when compared with other c values tested.

In the end, it can be said that choosing c is very important in minimizing the number of nodes required to obtain a desired degree of accuracy. As hypothesized, a good value is where the second derivative of $\Phi_Y(c)$ is closest to zero. This could be potentially found in closed form or numerically. However, numerical determination will also require computing time. Therefore, if the singularities of $\Phi_Y(s)$ are easily identifiable, then choosing c to be half the value of the real, positive singularity closest to the imaginary axis seems to work well. If a singularity does not exist or is not easily identifiable, finding the second derivative of $\Phi_Y(c)$ and determining where it is zero results in a good choice for c. The next best technique for finding good c values is by plotting $\Phi_Y(c)$ and graphically attempting to determine where the function is well behaved to choose c. This worked pretty well for the BPSK case in AWGN and with a phase error in the receiver. However, if such a value is strongly dependent on the SNR, a good value of c will be different for each SNR value and then numerically finding good values for c may be the easiest method. However, as previously stated, this will add to the computational time required to implement the method.

CHAPTER V

CONCLUSIONS

5.1 Conclusions

The technique described in [2] to analyze error probability in communication systems using the two-sided Laplace transform and a Gauss-Quadrature rule has been fully described. Implementation of several test cases has verified that the technique works for models of varying complexity. Methods for selecting the constant c have been explained and tested. These methods have been shown to provide accurate results when compared to analytically derived error probabilities for the cases considered. Additionally, the technique was implemented assuming equal *a priori* probabilities for each binary signal set. The method can still be used in the case where the *a priori* probabilities are not equal. Furthermore, this would not change the resulting Laplace transforms.

The most significant utilization of this technique applies to models where closed form expressions for the error probability are not known. Therefore, in many cases, numerical integration is required to obtain the error probability. As shown for the case of BPSK in AWGN with phase error in the receiver, the error probability obtained using this method resulted in significant time savings over numerically integrating the traditional error probability expression of (3.30) while obtaining similar results. Therefore, it is

suggested that when a closed form error probability can not be found analytically, this method may allow error probabilities to be computed faster.

5.2 Suggestions for Further Research

An avenue of further research would be to apply this technique to cases where $\Phi_Y(s)$ is not analytically known. As suggested in [2], the method can still be used if only a few numerical values of $\Phi_Y(s)$ are known. Investigating this has the potential to allow analysis of error performance if a closed form expression for $\Phi_Y(s)$ is not obtainable.

Also, further application of this technique to models of higher complexity shows good potential. For example, one suggestion is to apply this method in analyzing diversity reception of DPSK and NFSK. In [22], this method was successfully applied to analysis of the error performance of wideband direct-sequence spread-spectrum (DSSS).

Investigating the possible extension of this technique to models with M-ary signal sets may also prove to be useful. Specifically, using this method to analyze QPSK should be a relatively easy step. The effect of phase error at the receiver could be considered in a similar manner as that in the BPSK of this thesis. Additionally, one could consider possible cross-talk problems that arise in the system. The method could be further extended to M-ary PSK and other M-ary systems.

Finally, investigating the recursive implementation of this method would possibly reduce computational requirements. Recursive implementation seems possible if the nodes are doubled in each successive implementation. This is because the abscissa locations when doubling the nodes would be maintained. Additionally, implementing the method when there is no known error probability could be conducted. In such a situation,

the method could be recursively implemented until the error probability change between implementations falls below some set threshold. Another way to ensure the method is working would be to compare the results to those obtained through simulation.

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Appendix A

MATLAB PROGRAMS

This Appendix shows the code used to generate the various error probabilities for the different models explored in this paper. Commands used to plot the results are not displayed. Note that, with the exception of the antipodal case, the programs are implemented with loops. Further computational time would be saved if these loops were implemented in matrix form like for the antipodal case, but that was not the focus of my efforts.

A.1 Antipodal Baseband Signaling

The MATLAB program used to find the error probability in the antipodal baseband signaling model is considered here. The program is given below:

```
clear;
A=2; T_b=.5;        %Amp. of Signal; Bit Time
n=30; m_Y=A*T_b;  %# of Nodes to be Used; m_Y|+A Sent
k=1:n/2;           %Index for tau_k
tau_k=tan((2*k-1)*pi/n);
SNR_dB=1:15;
SNR=10.^(SNR_dB/10);
var_Y=A^2*T_b^2./(2*SNR); %Variance of Y
c=(A^2*T_b^2)./(2*var_Y); %Setting for c; equal to SNR
PHI=exp(((.5*var_Y.*c.^2)'*(1+j*tau_k).^2)-
   m_Y*c'*(1+j*tau_k));
x=ones(1,length(SNR));
   %Creates a Matrix of tau_k(s) to avoid using a loop to
   perfrom the summation for several SNR values
tau_k=(tau_k'*x)';
PE_GQ=(1/n)*sum((real(PHI)+tau_k.*imag(PHI))');
   %P_E using Gauss-Quad. Method
PE(SNR_dB)=.5*erfc(sqrt(SNR));
   %Known Closed Form Error Probability
```

A.2 BPSK in AWGN with Tikinov Phase Error

The program for finding the error probabilities for the BPSK case is shown here. The following part of the program shows general variables and constants used in both methods used:

```
clear;
A=2; Tb=.5;        %Amp. of Signal; Bit time
n=100; m_Y=A*Tb;  %# of Nodes; m_Y|+A Sent
k=1:n/2;          %Index for tau_k
tau_k=tan((2*k-1)*pi/n);
track=-10;
   %The amount of difference desired between tracking loop
   SNR (in dB)and AWGN SNR.  (Positive values mean tracking
   loop SNR is greater than the AWGN SNR)
SNRdB=1:15; SNR=10.^(SNRdB./10);
var_Y=(A^2*Tb^2)./(2*SNR); %Variance of Y given phi
rho=10.^((SNRdB+track)./10);
```

The next portion of the program shows numerical integration using MATLAB's *quad8* command.

```
tic
for i=1:length(SNR);
   PE_Q8(i)=quad8('petikinov',-pi,pi,[],[],rho(i),SNR(i));
end;
time1=toc;
```

The following shows the function definition of *petikinov* called above in the *quad8* command:

```
function y=petikinov(phi,rho,SNR)
y=(1./(4*pi*besseli(0,rho))).*exp(rho.*cos(phi)).*
      erfc(sqrt(SNR).*cos(phi));
```

Finally, the Gauss-Quadrature method is implemented as follows:

```
tic
for i=1:length(SNR);
   c=5;
   ww=c+j*c.*tau_k;
   PHI=(besseli(0,(-A*Tb*ww+rho(i)))./
      (besseli(0,rho(i)))).*exp(.5*var_Y(i)*ww.^2);
   PE_GQ(i)=(1/n)*sum(real(PHI)+tau_k.*imag(PHI));
end;
```

```
time2=toc;

SAVE=round(100-(time2/time1)*100);
   %Calculation of time savings
PE=.5.*erfc(sqrt(SNR));
   %Prob. of Error with phase error = 0
```

A.3 BPSK in AWGN with Rayleigh Fading

Again, the first portion of the program involves defining constants and variables

```
clear; Ac=1; Tb=1; n=50; k=1:n/2; tau_k=tan((2*k-1)*pi/n);
SNRdB=1:15; SNR=10.^(SNRdB/10); N0=Ac^2*Tb./(2*SNR);
c=linspace(.01,3,20);
```

The next portion of the program involved finding a good value for the constant c to obtain accurate results with the fewest possible nodes. This involves finding that portion of the function $\Phi_Y(c)$ that behaves nicely. This enables the Gauss-Quadrature method to approximate the function more accurately with a set amount of nodes or allows you to decrease the number of nodes used to any desired degree of accuracy and reduce computation time. The code is:

```
tic;
phic=[];
for jj=1:length(SNRdB);
   for ii=1:length(c);
      s=c(ii);
      xx=s^2/2;
      yy=quad8('fade',0,1,[],[],xx);
      uu=1-exp(s^2/4)*(s*sqrt(pi/4)-(s^2/2)*yy);
      phic(length(c)*(jj-1)+ii)=exp(.5*1/(SNR(jj))*Tb*s.^2).*uu;
   end;
end;
phic=reshape(phic,length(c),length(SNRdB));
[YY,opt]=min(abs(diff(phic,2)));
c_opt=c(opt);
time1=toc;
```

The function call to *fade* in the *quad8* command calls the function that defines the argument of the numerical integration required in this model (reference (3.54)). The code in this file is:

```
function y=fade(z,w)
y=exp(-.5*w*z.^2);
```

Finally, the Gauss-Quadrature method is applied as follows:

```
tic;
for ii=1:length(N0);
   %s=c+j*c*tau_k;
   s=c_opt(ii)+j*c_opt(ii)*tau_k;
   xx=s.^2/2;
   for jj=1:length(xx);
      yy(jj)=quad8('fade',0,1,[],[],xx(jj));
   end;
   uu=s*sqrt(pi/4)-((s.^2)/2).*yy;
   PHI=exp(.5*N0(ii)*Tb*s.^2).*(1-exp(s.^2/4).*uu);
   PE_GQ(ii)=(1/n)*sum(real(PHI)+tau_k.*imag(PHI));
   clear s;
end;
time2=toc;

PE=.5*erfc(sqrt(SNR));
PE_T=.5*(1-sqrt(SNR./(1+SNR))); %Theoretical, closed form error
probability

DIFFERENCE=(1/100)*round(max(abs(PE_GQ-
PE_T)./(.5*(PE_GQ+PE_T))*100)*100);
      %Calculates the percent difference
```

A.4 NFSK in AWGN with Rayleigh Fading

Finally, the NFSK program is presented. The first portion of the program again deals with declaring the various constants and variables used in the program.

```
clear;
n=10;  %# of Nodes
k=1:n/2;  %Index for tau_k
tau_k=tan((2*k-1)*pi/n);  %Location of abscissas
SNRdB=1:20;
SNR=10.^(SNRdB/10);
```

Next, the theoretical error probability in the case of an AWGN channel (PE) and the close form solution for the AWGN channel with Rayleigh fading (PE_FADE1) are computed.

```
PE=.5*exp(-.5*SNR);
PE_FADE1=.5*(1./(1+.5*SNR));
```

The Gauss-Quadrature method is now applied with the constant *c* set to half of the value of the smallest positive singularity.

```
for i=1:length(SNR);
   c=.25; %Sets the optimum Value of c
   ww=c+j*c.*tau_k;
   A=(1/(SNR(i)))-(1./(2*(.5+ww)))+1;
   B=4*SNR(i)*(.5-ww).*(.5+ww);
   PHI=1./(A.*B);
   PE_GQ(i)=(1/n)*sum(real(PHI)+tau_k.*imag(PHI));
end;

ERROR=100*abs((PE_FADE1-PE_GQ))./(.5*(PE_FADE1+PE_GQ));
MAX_ERROR=round(100*max(ERROR))/100;
MAX_ERROR=num2str(MAX_ERROR);
```

Appendix B

INTEGRAL DERIVATIONS

This Appendix deals with an integral used in the text of the paper in Chapter III.

B.1 Integral for BPSK in AWGN and Rayleigh Fading. This case required an integral of the form

$$\int_0^\infty u \exp\left[-au^2 - 2bu\right] du \tag{B.1}$$

to be solved. The first step in finding this integral is to complete the square in the exponent to yield

$$\int_0^\infty u e^{\left[-a\left(u^2 + 2b/_a u + b^2/_a - b^2/_a\right)\right]} du = e^{\frac{b^2}{a}} \int_0^\infty u e^{\left[-a\left(u + b/_a\right)^2\right]} du \tag{B.2}$$

A change of variables of $z = u + b/a$ is then applied to (B.2) to yield

$$e^{b^2/_a} \int_0^\infty \left(z - b/_a\right) e^{\left[-az^2\right]} dz = e^{b^2/_a} \int_{b/_a}^\infty z e^{-az^2} dz - \frac{b}{a} e^{b^2/_a} \int_{b/_a}^\infty e^{-az^2} dz . \tag{B.3}$$

Considering the first integral in (B.3) and accomplishing another change of variables of $w = az^2$ results in the first integral being

$$e^{b^2/_a} \int_{b/_a}^\infty z e^{-az} dz = \frac{1}{2a} e^{b^2/_a} \int_{b^2/_a}^\infty e^{-w} dw = \frac{1}{2a} e^{b^2/_a} e^{-b^2/_a} = \frac{1}{2a} . \tag{B.4}$$

Next, consider the second integral of (B.3). This could easily be represented in the form of a complementary error function but, since the limit is complex, MATLAB could not implement it. Therefore, the second integral is put in the form

$$\begin{aligned} -\frac{b}{a}e^{b^2/a}\int_{b/a}^{\infty} e^{-az^2}dz &= -\frac{b}{a}e^{b^2/a}\left[\int_{b/a}^{\infty} e^{-az^2}dz + \int_{0}^{b/a} e^{-az^2}dz - \int_{0}^{b/a} e^{-az^2}dz\right] \\ &= -\frac{b}{a}e^{b^2/a}\left[\int_{0}^{\infty} e^{-az^2}dz - \int_{0}^{b/a} e^{-az^2}dz\right] \end{aligned} \tag{B.5}$$

Applying the integral

$$\int_0^{\infty} e^{-ay^2}dy = \frac{1}{2}\sqrt{\frac{\pi}{a}} \tag{B.6}$$

from [18] to the first integral in (B.5) and making a change of variables of $x = (a/b)z$ to the second integral results in

$$-\frac{b}{a}e^{b^2/a}\int_{b/a}^{\infty} e^{-az^2}dz = -\frac{b}{a}e^{\left(b^2/a\right)}\left[\frac{1}{2}\sqrt{\frac{\pi}{a}} - \frac{b}{a}\int_0^1 e^{-\left(b^2/a\right)x^2}dx\right]. \tag{B.7}$$

Finally, the original integral is expressed in the form

$$\int_0^{\infty} ue^{\left[-au^2-2bu\right]}du = \frac{1}{2a} - \frac{b}{a}e^{\left(b^2/a\right)}\left[\frac{1}{2}\sqrt{\frac{\pi}{a}} - \frac{b}{a}\int_0^1 e^{-\left(b^2/a\right)x^2}dx\right]. \tag{B.8}$$